Math. 311 (Fulling) 26 October 1998

Test B – Solutions

1. *(18 pts.)* For each of these sets, tell whether or not it is a *basis* for $\mathbf{R}^3$; and if not, explain briefly why not.

   (a) $\{(1,1,1),(0,1,1),(0,0,2)\}$

YES — From the matrix $\begin{pmatrix}1&1&1\\0&1&1\\0&0&2\end{pmatrix}$ it is obvious that these vectors are independent. Alternatively, the determinant equals $2\neq 0$.

   (b) $\{(3,5,9),(0,6,-20)\}$

NO — There are not enough vectors to span a 3-dimensional space.

   (c) $\{(1,0,1),(0,1,1),(1,2,3)\}$

NO — Form a matrix and reduce:

$$\begin{pmatrix}1&0&1\\0&1&1\\1&2&3\end{pmatrix}\to\begin{pmatrix}1&0&1\\0&1&1\\0&0&0\end{pmatrix}.$$

We see that the vectors are not independent (and don't span) because the third vector equals the first plus twice the second. Or, you can calculate the determinant and get 0 .

2. *(10 pts.)* Let $\mathcal{V}$ be the vector space spanned by the functions $\{\cos t,\cos 2t,\sin t,\sin 2t\}$. Let $\mathcal{S}$ be the space of solutions of the differential equation $y''+y=0$. You may rest assured that $\mathcal{S}$ is a sub*set* of $\mathcal{V}$. Is $\mathcal{S}$ a sub*space* of $\mathcal{V}$? Explain!

YES — Use any one of these 5 (really only 2) arguments:

1. $\mathcal{S}$ is the span of $\{\cos t,\sin t\}$, and the span of a set is always a subspace of the larger vector space involved.

1′. The solutions are the functions $c_1\cos t+c_2\sin t$, and we can show directly that this set is closed under addition and scalar multiplication:

$$r(c_1\cos t+c_2\sin t)+(d_1\cos t+d_2\sin t)=(rc_1+d_1)\cos t+(rc_2+d_2)\sin t.$$

2. The solution space of a homogeneous linear equation is always a subspace.

2′. $\mathcal{S}$ is the kernel of the linear operator $L\equiv\frac{d^2}{dt^2}+1$ (that is, $L(y)\equiv y''+y$), and the kernel of a linear operator is always a subspace.

2″. We can show by brute force that $\mathcal{S}$ is closed under addition and scalar multiplication: If $y_1''+y_1=0$ and $y_2''+y_2=0$, then

$$(ry_1+y_2)''+(ry_1+y_2)=r(y_1''+y_1)+(y_2''+y_2)=0.$$

3. *(12 pts.)* Find the matrix M that changes *coordinates* from the natural basis in $\mathbf{R}^2$ to the basis

$$\mathcal{B} = \left\{ \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \equiv \{\vec{b}_1, \vec{b}_2\}.$$

Let

$$B = \begin{pmatrix} \top & \top \\ \vec{b}_1 & \vec{b}_2 \\ \bot & \bot \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}.$$

B maps *coordinates* from $\mathcal{B}$ to the natural basis. Therefore, what we want is

$$M = B^{-1} = \frac{1}{(-2)} \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}.$$

Alternative argument: We have

$$\vec{b}_1 = 0\hat{e}_1 + 2\hat{e}_2\,,$$
$$\vec{b}_2 = 1\hat{e}_1 + 1\hat{e}_2\,.$$

That is, $\begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$ is the matrix that expresses the $\mathcal{B}$ basis vectors in terms of the natural basis. Therefore, its transpose (which is what we called B above) is the matrix that expresses the natural coordinates in terms of the coordinates with respect to basis $\mathcal{B}$. Again, the desired M is the inverse of that.

4. *(25 pts.)* Suppose that the linear function $L\colon \mathbf{R}^2 \to \mathbf{R}^3$ is represented by $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ with respect to the natural basis.

   (a) What is the rank of A? <u>2</u> (since there are two independent columns).

   (b) Find the matrix representing f if the basis in the domain is changed to $\mathcal{B}$ (see Question 3), the basis for the codomain remaining unchanged.

First map the coordinates from $\mathcal{B}$ to the natural coordinates, then apply A:

$$AM^{-1} = AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 8 & 7 \end{pmatrix}.$$

   (c) Find the matrix representing f if the basis $\mathcal{B}$ is used for both domain and codomain.

Same as (b), except that at the end we need to map the coordinates of the output vector from natural to $\mathcal{B}$:

$$MAM^{-1} = B^{-1}AB = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 8 & 7 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 4 & 3 \end{pmatrix}.$$

5. *(Essay – 15 pts.)* Explain how the "superposition principles" are used in solving a nonhomogeneous differential equation such as $\frac{d^2y}{dt^2} + 4y = \cos(3t)$.

Best student response received:

> 1st apply superposltion principle #1 which states that the linear combination of solution to a homogeneous problem is a solution to that problem. From this we have $L(y_1) = 0$ and $L(y_2) = 0$, where y_1 and y_2 are 2 solutions and when added make up a solution to the homogeneous equation. So in general $y = c_1y_1 + c_2y_2$. Now we apply the superposition principle #2, which states that the sum of a particular solution of a nonhomogeneous and the solution of its corresponding homogeneous solution, give a solution of the original nonhomogeneous problem. So if $L(y_p) = f$, then a solution to the nonhomogeneous differential equation to problem 5 is
> $$y = y_p + c_1y_1 + c_2y_2 .$$

To this I would add only that the solutions y_1 and y_2 must be chosen to be linearly independent, and that the 3rd superposition principle (difference of two solutions of the same nonhomogeneous problem is a solution of the homogeneous one) proves that the final formula gives *all* the solutions.

6. *(20 pts.)* Let $\mathcal{V}$ be the span of the functions $\{\vec{v}_1 = e^{-t}, \vec{v}_2 = te^{-t}, \vec{v}_3 = t^2e^{-t}\}$ (that is, each element of $\mathcal{V}$ is e^{-t} times an element of $\mathcal{P}_2$). Let $L: \mathcal{V} \to \mathcal{V}$ be the differentiation operator with $\mathcal{V}$ as both domain and codomain (that is, $L: \mathcal{V} \to \mathcal{V}$ and $Lf \equiv f'$).

(a) Find the matrix representing L with respect to the basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

Calculate

$$L(\vec{v}_1) = -e^{-t} = -\vec{v}_1 , \quad L(\vec{v}_2) = e^{-t} - te^{-t} = \vec{v}_1 - \vec{v}_2 , \quad L(\vec{v}_3) = 2te^{-t} - t^2e^{-t} = 2\vec{v}_2 - \vec{v}_3 .$$

Put the coefficients in these equations into the columns of the matrix in the appropriate order:

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{pmatrix} .$$

(b) Is L surjective (onto)? If not, what is its range?

YES — Columns span the whole space.

(c) Is L injective (one-to-one)? If not, what is its kernel?

YES — Matrix is nonsingular, kernel is just $\{\vec{0}\}$. (Note also that (c) follows from (b) or vice versa by the theorem (rank) + (nullity) = (dimension of domain).)