Math. 311 (Fulling) 13 February 2002

Test A – Solutions

Name: ____

Calculators may be used for simple arithmetic operations only!

1. *(12 pts.)* Find the inverse (if it exists) of the matrix $M = \begin{pmatrix} 3 & 8 \\ 1 & 3 \end{pmatrix}$.

Reduce the augmented matrix:

$$\begin{pmatrix} 3 & 8 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{pmatrix} \xrightarrow{[1]\leftrightarrow[2]} \begin{pmatrix} 1 & 3 & 0 & 1 \\ 3 & 8 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow{[2]\to[2]-3[1]} \begin{pmatrix} 1 & 3 & 0 & 1 \\ 0 & -1 & 1 & -3 \end{pmatrix} \xrightarrow[{[2]\to-[2]}]{[1]\to[1]+3[2]} \begin{pmatrix} 1 & 0 & 3 & -8 \\ 0 & 1 & -1 & 3 \end{pmatrix}.$$

Therefore,

$$M^{-1} = \begin{pmatrix} 3 & -8 \\ -1 & 3 \end{pmatrix}.$$

It is easy to check that $MM^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

2. *(10 pts.)* A function $f(x,y)$ satisfies the equations $\dfrac{\partial f}{\partial x} = -\dfrac{2x}{y} f$, $\quad \dfrac{\partial f}{\partial y} = \dfrac{x^2}{y^2} f$. Calculate $\dfrac{d}{dx} f(x,x^2)$.

$$\frac{d}{dx} f(x,x^2) = \nabla f \cdot \frac{d}{dx} \begin{pmatrix} x \\ x^2 \end{pmatrix} = \left.\frac{\partial f}{\partial x}\right|_{y=x^2} \times 1 + \left.\frac{\partial f}{\partial y}\right|_{y=x^2} \times (2x)$$

$$= -\frac{2x}{x^2} f + \frac{x^2}{x^4} (2x) f = \left(-\frac{2}{x} + \frac{2}{x}\right) f = 0.$$

Remark: $f(x,y) = e^{-x^2/y}$ is a function with these properties.

3. *(18 pts.)* A curve C in three-dimensional space is specified by the parametric equations

$$x = t, \quad y = t\sin t, \quad z = \cos t.$$

(a) Find the tangent vector to C at the point where $t = \pi$.

Let $\vec r(t) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Then $\vec r\,'(t) = \begin{pmatrix} 1 \\ \sin t + t\cos t \\ -\sin t \end{pmatrix}$, so $\vec r\,'(\pi) = \begin{pmatrix} 1 \\ -\pi \\ 0 \end{pmatrix}$.

(b) Find the directional derivative of $f(x,y,z) = x + ze^y$ at that point, in the direction of the curve.

The unit vector in the direction of the curve is $\vec{r}\,'(\pi)$ divided by its length:

$$\hat{u} = \frac{1}{\sqrt{1+\pi^2}} \begin{pmatrix} 1 \\ -\pi \\ 0 \end{pmatrix}. \qquad \text{Also,} \quad \vec{r}(\pi) = \begin{pmatrix} \pi \\ 0 \\ -1 \end{pmatrix}.$$

So

$$\nabla f = (1, ze^y, e^y)\Big|_{\vec{r}(\pi)} = (1, -1, 1).$$

Thus

$$\frac{\partial f}{\partial \hat{u}} = \nabla f \cdot \hat{u} = \frac{1+\pi}{\sqrt{1+\pi^2}}\,.$$

4. *(15 pts.)* Producing a refrigerator requires 0.1 ton of steel and 0.2 ton of plastic. Producing an airplane requires 5 tons of steel and 2 tons of plastic. Producing a ton of steel consumes 3 tons of coal and 10 barrels of water. Producing a ton of plastic consumes 2 tons of coal and 50 barrels of water. Organize these facts into matrices, and find the matrix that tells you how much coal (c) and water (w) is needed to make r refrigerators and a airplanes.

Let s and p be the quantities of steel and plastic, and let

$$\begin{pmatrix} s \\ p \end{pmatrix} = B \begin{pmatrix} r \\ a \end{pmatrix} = \begin{pmatrix} 0.1 & 5 \\ 0.2 & 2 \end{pmatrix} \begin{pmatrix} r \\ a \end{pmatrix}, \qquad \begin{pmatrix} c \\ w \end{pmatrix} = A \begin{pmatrix} s \\ p \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 10 & 50 \end{pmatrix} \begin{pmatrix} s \\ p \end{pmatrix}.$$

Then $\begin{pmatrix} c \\ w \end{pmatrix} = AB \begin{pmatrix} r \\ a \end{pmatrix}$, where

$$AB = \begin{pmatrix} 3 & 2 \\ 10 & 50 \end{pmatrix} \begin{pmatrix} 0.1 & 5 \\ 0.2 & 2 \end{pmatrix} = \begin{pmatrix} 0.7 & 19 \\ 11 & 150 \end{pmatrix}.$$

5. *(10 pts.)* Classify each of these integral operators as linear, affine, or fully nonlinear (as a function of g). (g is an element of $\mathcal{C}(0,1)$ — that is, a function.)

(a) $A(g) = \displaystyle\int_0^t e^{(t-s)} g(s)\, ds\,.$ ($A(g)$ is another element of $\mathcal{C}(0,1)$ — a function of the variable t . In other words, $A\colon \mathcal{C}(0,1) \to \mathcal{C}(0,1)$.)

Linear. This is clear from the form of the integrand; or, one can easily verify that

$$A(\lambda g + h) = \int_0^t e^{(t-s)} [\lambda g(s) + h(s)]\, ds = \lambda \int_0^t e^{(t-s)} g(s)\, ds + \int_0^t e^{(t-s)} h(s)\, ds = \lambda A(g) + A(h).$$

(b) $B(g) = \int_0^1 t e^{g(t)}\, dt$. ($B(g)$ is an element of $\mathbf{R}$ — a number. $B\colon \mathcal{C}(0,1) \to \mathbf{R}$.)

Nonlinear. Again, this is pretty obvious because the g is up in the exponent. A formal counterexample (to the homogeneity clause of the definition) is

$$B(\lambda g) = \int_0^1 t e^{\lambda g(t)}\, dt \neq \int_0^1 t\lambda e^{g(t)}\, dt = \lambda B(g).$$

(B is not affine, because it's not of the form of a linear operator plus a fixed vector (which would be a constant number in this case). An example of an affine operator $C\colon \mathcal{C}(0,1) \to \mathcal{C}(0,1)$ is $C(g)(t) = A(g)(t) + \cos t$, A as in part (a).)

6. *(10 pts.)* Construct the best affine approximation (also known as the first-order approximation) to $T(x,y) = \begin{pmatrix} \sqrt{x^2+4y^2} \\ x-y \end{pmatrix}$ in the neighborhood of the point $(x_0, y_0) = (1,1)$.

The matrix of partial derivatives is

$$JT = \begin{pmatrix} \frac{x}{\sqrt{x^2+4y^2}} & \frac{4y}{\sqrt{x^2+4y^2}} \\ 1 & -1 \end{pmatrix}, \qquad \text{so} \quad J_{\vec{r}_0} T = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{4}{\sqrt{5}} \\ 1 & -1 \end{pmatrix}.$$

Therefore,

$$T(x,y) \approx T(x_0,y_0) + J_{\vec{r}_0} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = \begin{pmatrix} \sqrt{5} \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{4}{\sqrt{5}} \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x-1 \\ y-1 \end{pmatrix}.$$

7. *(25 pts.)* Find all solutions (x,y,z) of $\left\{ \begin{aligned} y - \ z &= 1, \\ x - \ y - 2z &= 0, \\ 2x + 3y + Az &= B. \end{aligned} \right\}$ (A and B are arbitrary, but fixed, parameters. Certain special values of A and B will require special attention.)

$$\begin{pmatrix} 0 & 1 & -1 & 1 \\ 1 & -1 & -2 & 0 \\ 2 & 3 & A & B \end{pmatrix} \xrightarrow{[2]\leftrightarrow[1]} \begin{pmatrix} 1 & -1 & -2 & 0 \\ 0 & 1 & -1 & 1 \\ 2 & 3 & A & B \end{pmatrix}$$

$$\xrightarrow{[3]\to[3]-2[1]} \begin{pmatrix} 1 & -1 & -2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 5 & A+4 & B \end{pmatrix} \xrightarrow[1\to[1]+[2]]{[3]\to[3]-5[2]} \begin{pmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & A+9 & B-5 \end{pmatrix}.$$

Case I: If $A = -9$ and $B \neq 5$, there are no solutions, because the bottom row gives an inconsistent equation.

Case II: If $A = -9$ and $B = 5$, then z is an arbitrary parameter and

$$y = z + 1, \qquad x = 3z + 1.$$

Case III: If $A \neq -9$, continue reducing:

$$\xrightarrow{[3]\to[3]/(A+9)} \begin{pmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & \frac{B-5}{A+9} \end{pmatrix} \xrightarrow[{[2]\to[2]+[3]}]{[1]\to[1]+3[3]} \begin{pmatrix} 1 & 0 & 0 & 1+3C \\ 0 & 1 & 0 & 1+C \\ 0 & 0 & 1 & C \end{pmatrix},$$

where we define $C = \dfrac{B-5}{A+9}$ to save writing. Therefore, in this case there is the unique solution

$$x = 1 + 3C, \qquad y = 1 + C, \qquad z = C.$$