Math. 311 (Fulling) 20 March 2002

Test B – Solutions

Name: ______________________________

Calculators may be used for simple arithmetic operations only!

1. *(24 pts.)* Let $L\colon \mathcal{P}_2 \to \mathcal{P}_2$ be the differential operator $(Lp)(t) \equiv p''(t) + 2\,t\,p'(t)$.

(a) Find the matrix that represents L with respect to the standard basis $\{t^2, t, 1\}$ for $\mathcal{P}_2$.

Method 1: Calculate $L(t^2) = 4t^2 + 2$, $L(t) = 2t$, $L(1) = 0$. Then by the kth-column rule, the matrix is

$$M = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

Method 2: Calculate $L(at^2 + bt + c) = 4at^2 + 2bt + 2a$. So we must have

$$\begin{pmatrix} 4a \\ 2b \\ 2a \end{pmatrix} = M \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

The conclusion is the same as before.

(b) Find the kernel of L. Is L injective?

Method 1: Find the kernel of M: M row-reduces to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so the solutions of the homogeneous equation are $a = 0$, $b = 0$, c arbitrary. Translate back into terms of polynomials: The kernel of L consists of the constant functions (the polynomial space $\mathcal{P}_0$). This is a nontrivial subspace, so L is not injective.

Method 2: Following Method 2 for part (a), we see that $p \in \ker L$ means that $4a = 0$, $2b = 0$, and $2a = 0$. Again this means that $a = b = 0$ but c is unrestricted.

Method 3: Solve the differential equation $p'' + 2tp' = 0$ directly: Let $q = p'$ so that we can deal with a first-order equation, $q' = -2tq$. It is separable:

$$\begin{aligned} \frac{dq}{q} = -2tdt \;&\Rightarrow\; \ln q = -t^2 + c \\ &\Rightarrow\; q = Ce^{-t^2} \\ &\Rightarrow\; p = C\int e^{-t^2} + D. \end{aligned}$$

The terms with $C \neq 0$ are not polynomials and hence are irrelevant. The only polynomial solutions are the constant functions.

(c) Find the range of L. Is L surjective?

Method 1: The range is represented by the span of the columns of M, which is all vectors in $\mathbf{R}^3$ of the form

$$A\begin{pmatrix}2\\0\\1\end{pmatrix}+B\begin{pmatrix}0\\1\\0\end{pmatrix}.$$

In terms of polynomials, the range of L consists of all polynomials of the form $C(2t^2+1)+Bt$. This is not the entire space $\mathcal{P}_2$, so L is not surjective.

Method 2: Suppose that At^2+Bt+C is in the range. Then by Method 2 for (a) we must have

$$A=4a, \quad B=2b, \quad C=2a.$$

It is easy to see that this system can be solved for (a,b,c) precisely when $A=2C$. The conclusion is as before.

Method 3 (because an honors test should be challenging for the teacher, too): An integrating factor for the first-order nonhomogeneous equation $q'+2tq=f(t)$ is the reciprocal of a solution of the corresponding homogeneous equation: $\mu(t)=e^{+t^2}$. Proceed:

$$\mu f = e^{t^2}q' + 2te^{t^2}q = \frac{d}{dt}\left(e^{t^2}q\right),$$

$$q(t)=e^{-t^2}\int^t e^{s^2}f(s)\,ds.$$

We are interested only in the cases where f is a second-degree polynomial, and we need to find out in which such cases q turns out to be a first-degree polynomial (so that its antiderivative, p, will be in $\mathcal{P}_2$). If $f(s)$ is proportional to s, the integral is elementary and turns out (for a particular choice of the constant of integration) to be proportional to e^{t^2}; thus q is a constant. This shows that vectors of the form Bt are in the range. If f is a constant, it is notorious that the integral ("the error function") is not an elementary function; thus q can't be a polynomial, and the constants C are not in the range (and hence neither are vectors $Bt+C$ with $C\neq 0$). It remains see whether t^2, or t^2 plus some constant, is in the range. Let's see what the substitution $w=s^2$ ($ds=dw/2\sqrt{w}$) does to the integrals:

$$\int e^{s^2}s^2\,ds=\frac{1}{2}\int e^w w^{1/2}\,dw, \qquad \int e^{s^2}\,ds=\frac{1}{2}\int e^w w^{-1/2}\,dw.$$

Evaluate the second integral by parts, with $u=e^w$, $v=w^{1/2}$:

$$\int e^{s^2}\,ds = e^w w^{1/2} - \int e^w w^{1/2}\,dw.$$

Thus

$$\int^t e^{s^2}\,ds = e^{t^2}t - 2\int^t e^{s^2}s^2\,ds.$$

Since the integral on the left is not elementary, the one at the end can't be elementary either. However,

$$\int^t e^{s^2}(2s^2+1)\,ds = e^{t^2}t$$

is elementary, and the equation $q'+2tq=2t^2+1$ therefore has the solution $q(t)=t$. Thus the original equation $p''+2tp'=2t^2+1$ has quadratic polynomial solutions $p(t)=\frac{1}{2}t^2+\text{constant}$!

2. *(20 pts.)* Let $M = \begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 6 & 1 \end{pmatrix}$ be the matrix of a linear function $L\colon \mathbf{R}^2 \to \mathbf{R}^3$ (with respect to the natural bases).

(a) What is the rank of M ?

2 , because the columns are obviously independent.

(b) What is the nullity of M (dimension of the kernel of L)?

Method 1: (dimension of domain) − (dimension of range) $= 2 - 2 = 0$.

Method 2: M row-reduces to

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

(the augmented matrix has another column of zeros at the end), so the only solution of the homogeneous equations is the zero vector. The kernel is $\{0\}$, which has dimension 0 .

(c) What matrix represents L when the basis $\left\{ \vec{b}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \vec{b}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ is used for the domain?

Put the new basis vectors together a matrix $U = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$. Clearly U maps coordinates with respect to the $\vec{b}$ basis into natural coordinates, ready to be acted upon by M . Thus the desired matrix is

$$MU = \begin{pmatrix} 5 & -1 \\ 3 & 3 \\ 8 & 5 \end{pmatrix}.$$

(Equivalently, construct this matrix by the kth-column rule, calculating $M\vec{b}_1$ and $M\vec{b}_2$ to be its columns.)

3. *(16 pts.)* It is well known that the two sets of functions

$$\{f_1 = 1,\ f_2 = \cos t,\ f_3 = \cos^2 t\} \quad \text{and} \quad \{g_1 = 1,\ g_2 = \cos t,\ g_3 = \cos(2t)\}$$

span the same vector space, $\mathcal{U}$, because $\cos(2t) = 2\cos^2 t - 1$.

(a) If r_j and s_j are defined by $y(t) = \sum_{j=1}^{3} r_j f_j = \sum_{j=1}^{3} s_j g_j$, find the matrix N such that

$$\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = N \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}.$$

Method 1: We have $g_1 = f_1$, $g_2 = f_2$, $g_3 = -f_1 + 2f_3$. Thus the matrix

$$K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix}$$

maps the f basis vectors to the g basis vectors. By a general principle, the transpose of K maps the g coefficients into the f coefficients, which is precisely what we want:

$$N = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Method 2: $\sum_{j=1}^{3} s_j g_j = s_1 + s_2 \cos t + s_3(2\cos^2 -1) = (s_1 - s_3)1 + s_2 \cos t + 2s_3 \cos^2 t$, so

$$r_1 = s_1 - s_3\,, \quad r_2 = s_2\,, \quad r_3 = 2s_3\,.$$

From this we read off the same N as in Method 1.

(b) If Q is the matrix of a linear function $L\colon \mathcal{U} \to \mathcal{U}$ with respect to the basis $\{f_j\}$, what matrix represents L with respect to the basis $\{g_j\}$? (Numerical calculations are not called for — just a formula in terms of N and Q.)

$N^{-1}QN$.

4. *(Essay — 20 pts.)* Consider the two problems

$$\left.\begin{aligned} x - \ y &= 1\,, \\ 2x - 2y &= 2\,; \end{aligned}\right\} \tag{1}$$

$$\frac{d^2y}{dt^2} = \sin(2t), \quad y(0) = 0, \quad y(\pi) = 0. \tag{2}$$

Discuss the analogy between these problems, and what the principles of linear algebra tell us about their solutions. *Vocabulary hints:* linear, homogeneous, affine, subspace, kernel, range, superposition, ...

This question did not work out as well as planned, because the two problems are not actually as "analogous" as I intended them to be. In (2) I should have either left off the boundary conditions, or replaced the differential equation by $\frac{d^2y}{dt^2} + y = \sin(2t)$; in either case the related homogeneous problem would then have nontrivial solutions, as (1) does.

I gave up to 5 points extra credit for actually solving the problems instead of just discussing them in generalities. Here is a good student response:

- both problems are linear & nonhomogeneous
  - Since they are both linear, we can use principles of superposition to determine their solutions. FOR PROBLEM 1, the kernel of the associated matrix is nontrivial ([the matrix is not] injective) so the associated homogeneous problem has a solution, $\vec{x}_\mathrm{h}$. We would then find the solutions to the nonhomogeneous problem and add these together to get the entire solution (superposition principle 2).
  - SIMILARLY FOR PROBLEM 2, the kernel for the homogeneous problem is nontrivial, and we would find that to yield y_h. If there is more than one solution to the homogeneous problem we can add these up to get the entire homogeneous solution (superposition principle 1). AND AGAIN we would find the particular solution[] of the type $\alpha \sin(2t)$, which we call y_p. The entire solution for $y(t)$ can be found by summing y_h & y_p (SUPERPOSITION PRINCIPLE 2). There is enough info, however (in the boundary conditions), so that the $y_\mathrm{h} = 0$ & y_p is unique.

[• The solutions are:]

(1) $x = y + 1$

(2) $y(t) = -\frac{1}{4}\sin(2t)$

5. *(20 pts.)* The 2×2 matrices form a vector space, $\mathcal{V}$.

(a) What is the dimension of $\mathcal{V}$?

4, the number of elements (parameters) in such a matrix. (The four matrices containing one 1 and three zeros form a natural basis for this space.)

(b) Here is a set of vectors from $\mathcal{V}$:

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

Decide whether this set is linearly independent; if it isn't, find an independent set with the same span.

It is independent. (A proof was not demanded, but you could imagine the matrices as rows

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

and reduce; or just notice that

$$a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + c \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

can't be zero unless all the coefficients are zero.)

(c) Show that the *antisymmetric* 2×2 matrices form a *subspace* of $\mathcal{V}$, and find its dimension.

Method 1: A matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is antisymmetric if it equals $-\begin{pmatrix} a & c \\ b & d \end{pmatrix}$; this requires $a = 0 = d$ and $c = -b$. Thus the antisymmetric matrices are the multiples of the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. This is obviously a subspace (being a span), and its dimension is 1.

Method 2: The condition of antisymmetry is $M^{\mathrm{t}} = -M$. Taking the transpose is a linear operation, so this is a homogeneous linear equation and its solutions form a subspace. To find the dimension we need to solve the equation, which is just repeating the first part of Method 1; there is only one independent parameter (b) in the solution, so the dimension is 1.

(d) *(extra credit)* Make a remark relating the three parts of this problem. *Hint:* Think about the *symmetric* matrices.

The span of the three matrices in (b) is the subspace of all 2×2 symmetric matrices, which thus has dimension 3. From an old homework problem we know that every square matrix is the sum of a symmetric and an antisymmetric part, and that the only matrix that is both symmetric and antisymmetric is 0. Thus the whole space is the *direct sum* of the symmetric and antisymmetric subspaces; its dimension is the sum of their dimensions, $4 = 3 + 1$.