Nabla in Curvilinear Coordinates

Reference: M. R. Spiegel, *Schaum's Outline of ... Vector Analysis ...* , Chapter 7 (and part of Chap. 8). (Page references are to that book.)

Suppose that $\vec{A} = A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$ with respect to the usual basis of unit vectors in spherical coordinates. What is $\nabla \cdot \vec{A}$? It is *not* $\frac{\partial A_r}{\partial r} + \frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_\phi}{\partial \phi}$. The reason is that the spherical unit vectors themselves are functions of position, and their own derivatives must be taken into account. The correct formula is [p. 165, solution 63(b)]

$$\nabla \cdot \vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}\,.$$

Where did this come from?

To set the stage, let's recall that the most elementary local basis associated with a coordinate system is the tangent vectors

$$\left\{\frac{\partial \vec{r}}{\partial u_j}\right\} \qquad \left[\text{in the spherical case,} \quad \left\{\frac{\partial \vec{r}}{\partial r}, \frac{\partial \vec{r}}{\partial \theta}, \frac{\partial \vec{r}}{\partial \phi}\right\}\right].$$

The *metric tensor* is defined as the matrix of inner products

$$g_{ij} = \frac{\partial \vec{r}}{\partial u_i} \cdot \frac{\partial \vec{r}}{\partial u_j}\,.$$

This means that the formula for arc length is

$$ds^2 = \sum_{ij} g_{ij}\, du_i\, du_j \qquad \left[= dr^3 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2 \quad \text{in sphericals}\right].$$

(In this context $dv\, du = +du\, dv$, *not* $-du\, dv$ as in surface integrals.)

Orthogonal coordinates are the very important special cases where g is a diagonal matrix:

$$g = \begin{pmatrix} {h_1}^2 & 0 & 0 \\ 0 & {h_2}^2 & 0 \\ 0 & 0 & {h_3}^2 \end{pmatrix} \quad \text{(in three-dimensional cases).}$$

That is, by definition

$$h_j = \left\| \frac{\partial \vec{r}}{\partial u_j} \right\| ;$$

in the spherical case,

$$h_r = 1, \quad h_\theta = r, \quad h_\phi = r \sin\theta.$$

In orthogonal coordinates we can define the orthonormal basis of unit vectors,

$$\hat{u}_j = \frac{1}{h_j} \frac{\partial \vec{r}}{\partial u_j} \,.$$

The other standard basis vectors, the normal vectors ∇u_j, satisfy

$$\hat{u}_j = h_j \nabla u_j \,.$$

Now we turn to the *gradient*. Note that there is some ambiguity in the term "components of the gradient in spherical coordinates", because we have *three* natural spherical bases to choose among. The most useful choice for "everyday" physics is the orthonormal basis, which is obtained by a rotation from the Cartesian basis $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$. By the chain rule one sees, for instance, that

$$\frac{\partial f}{\partial \theta} = \sum_j \frac{\partial f}{\partial x_j} \frac{\partial x_j}{\partial \theta} = \nabla f \cdot \frac{\partial \vec{r}}{\partial \theta} = h_\theta \hat{\theta} \cdot \nabla f.$$

But since this basis is ON, this is equivalent to saying that the $\hat{\theta}$ component of ∇f is ${h_\theta}^{-1} \frac{\partial f}{\partial \theta}$. In general [p. 137],

$$\nabla f = \sum_j \frac{1}{h_j} \frac{\partial f}{\partial u_j} \hat{u}_j \,,$$

which in the spherical case is [p. 165, solution 63(a)]

$$\nabla f = \frac{\partial f}{\partial r}\,\hat{r} + \frac{1}{r}\,\frac{\partial f}{\partial \theta}\,\hat{\theta} + \frac{1}{r\sin\theta}\,\frac{\partial f}{\partial \phi}\,\hat{\phi}.$$

To treat the *divergence* and curl, first note that

$$\hat{u}_1 = \hat{u}_2 \times \hat{u}_3 = h_2 h_3 (\nabla u_2) \times (\nabla u_3)$$

(and the two obvious cyclic permutations of this formula). Now consider one term of the divergence:

$$\begin{aligned} \nabla \cdot (A_1 \hat{u}_1) &= \nabla \cdot (A_1 h_2 h_3 \nabla u_2 \times \nabla u_3) \\ &= \nabla(A_1 h_2 h_3) \cdot (\nabla u_2 \times \nabla u_3) + A_1 h_2 h_3 \nabla \cdot (\nabla u_2 \times \nabla u_3). \end{aligned}$$

But the last term is $\vec{0}$, because $\nabla \cdot (\vec{B} \times \vec{C}) = \vec{C} \cdot (\nabla \times \vec{B}) - \vec{B}(\nabla \times \vec{C})$ and $\nabla \times \nabla u_j = 0$. So

$$\begin{aligned} \nabla \cdot (A_1 \hat{u}_1) &= \nabla(A_1 h_2 h_3) \cdot (\nabla u_2 \times \nabla u_3) \\ &= \frac{1}{h_2 h_3} (\hat{u}_2 \times \hat{u}_3) \cdot \nabla(A_1 h_1 h_2) \\ &= \frac{\hat{u}_1}{h_1 h_3} \cdot \left[\frac{\partial}{\partial u_1}(A_1 h_2 h_3) \cdot \frac{1}{h_1} \hat{u}_1 + (\hat{u}_2 \text{ and } \hat{u}_3 \text{ terms}) \right], \end{aligned}$$

where the previous formula for the gradient has been used. Only the $\hat{u}_1$ term survives the dot product, so

$$\nabla \cdot (A_1 \hat{u}_1) = \frac{1}{h_1 h_2 h_3}\,\frac{\partial}{\partial u_1}(A_1 h_2 h_3).$$

Adding the two cyclic analogues of this term one gets [p. 150]

$$\nabla \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1}(A_1 h_2 h_3) + \frac{\partial}{\partial u_2}(A_2 h_3 h_1) + \frac{\partial}{\partial u_3}(A_3 h_1 h_2) \right].$$

In spherical coordinates this reduces to the formula given at the beginning.

Incidentally, the denominator in the divergence formula, $h_1 h_2 h_3 = \sqrt{\det g}$, is equal to the Jacobian determinant that occurs in the curvilinear volume formula. This fact is a consequence of Part 2 of the Volume Theorem on p. 340 of *Linearity*, where the metric tensor was introduced without a name.

Similarly [p. 150; p. 154 for spherical case], the *curl* is

$$\nabla \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{Vmatrix} h_1 \hat{u}_1 & h_2 \hat{u}_2 & h_3 \hat{u}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{Vmatrix}.$$

An alternative way of deriving the divergence and curl formulas [pp. 151–152] is based on the integral definitions of divergence and curl [p. 365 of *Linearity*]. The factors h_j show up in the relationships between coordinate increments (Δu_j) and physical lengths, areas, and volumes.