Math. 311 (Fulling) 23 April 2004

Test C – Solutions

Name: ______________________________

Calculators may be used for simple arithmetic operations only!

1. *(20 pts.)* Let $\vec{v}_1 = (1,1,1,1)$, $\vec{v}_2 = (1,-1,2,-1)$.

   (a) Find an orthonormal basis, $\{\hat{u}_1, \hat{u}_2\}$, for the two-dimensional subspace of $\mathbf{R}^4$ spanned by $\vec{v}_1$ and $\vec{v}_2$.

$\|\vec{v}_1\|^2 = 1+1+1+1 = 4$, so a normalized vector is

$$\hat{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{2}(1,1,1,1).$$

Now $\langle \hat{u}_1, \vec{v}_2\rangle = \frac{1}{2}(1-1+2-1) = \frac{1}{2}$, so the part of $\vec{v}_2$ parallel to $\vec{v}_1$ is

$$\vec{v}_{2\|} = \hat{u}_1 \langle \hat{u}_1, \vec{v}_2\rangle = \frac{1}{4}(1,1,1,1),$$

and hence the perpendicular part is

$$\vec{v}_{2\perp} = \vec{v}_2 - \vec{v}_{2\|} = (1,-1,2,-1) - (\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}) = \frac{1}{4}(3,-5,7,-5).$$

Next we have

$$\|\vec{v}_{2\perp}\|^2 = \frac{1}{16}(9+25+49+25) = \frac{108}{16}.$$

(Actually, I could have dropped the factor $\frac{1}{4}$, since it cancels out at the next step.) Therefore, the normalized vector orthogonal to $\hat{u}_1$ is

$$\hat{u}_2 = \frac{\vec{v}_{2\perp}}{\|\vec{v}_{2\perp}\|} = \frac{1}{\sqrt{108}}(3,-5,7,-5) = \frac{1}{6\sqrt{3}}(3,-5,7,-5).$$

   (b) Give a formula for P, the orthogonal projection operator onto that subspace. (That is, for any $\vec{v}$ in $\mathbf{R}^4$, $P(\vec{v})$ is the part of $\vec{v}$ "parallel" to the plane $\operatorname{span}\{\vec{v}_1, \vec{v}_2\}$.)

This is just the next step in a Gram–Schmidt construction:

$$P(\vec{v}) = \vec{v}_{\|} = \hat{u}_1\langle \hat{u}_1, \vec{v}\rangle + \hat{u}_2\langle \hat{u}_2, \vec{v}\rangle,$$

where $\hat{u}_1$ and $\hat{u}_2$ were found in (a).

2. *(30 pts.)* In the (x,y) plane define new coordinates (u,v) by $x = \frac{u}{2}$, $y = \frac{u^2}{4} + v$.

   (a) Find the tangent vectors to the coordinate curves (as functions of u and v).

$$\frac{\partial \vec{r}}{\partial u} = \begin{pmatrix} \frac{1}{2} \\ \frac{u}{2} \end{pmatrix}, \qquad \frac{\partial \vec{r}}{\partial v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Note: Since these vectors are not orthogonal, there is no point in normalizing them to unit length. The same remark applies to the vectors in (b).

(b) Find the normal vectors to the coordinate "surfaces" (which are actually curves in this two-dimensional case), as functions of u and v.

From (a), the Jacobian of the coordinate transformation and its inverse (by the 2×2 Cramer's rule) are

$$J = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{u}{2} & 1 \end{pmatrix}, \qquad J^{-1} = \frac{1}{1/2}\begin{pmatrix} 1 & 0 \\ -\frac{u}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -u & 1 \end{pmatrix}.$$

The standard normal vectors are the rows of J^{-1}:

$$\nabla u = (2,0), \qquad \nabla v = (-u,1).$$

(c) Evaluate $\iint v^2\,dx\,dy$ over the region bounded by the curves $v=0$, $u=2$, $v=1$, and $u=1$.

From J found in (b), we have $\det J = \frac{1}{2}$. So the integral is

$$\iint v^2 \det J\,du\,dv = \int_1^2 du \int_0^1 \frac{1}{2} v^2\,dv = \frac{1}{2}\,\frac{v^3}{3}\bigg|_0^1 = \frac{1}{6}.$$

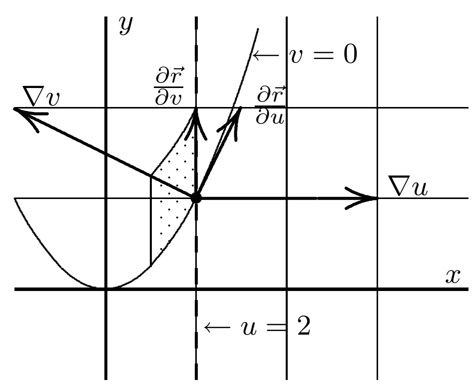
(d) Sketch the curves $v=0$ and $u=2$, the region in (c), and the two sets of basis vectors in (a) and (b) evaluated (and drawn) at the point $(u,v)=(2,0)$. **Clearly label the vectors as ∇u, $\frac{\partial\vec r}{\partial u}$, etc.**

The Cartesian coordinates of the point are $(x,y)=(1,1)$.
The $u=$ constant curves are vertical lines; the $v=$ constant curves are parabolas, $y=x^2+v$.

$$\frac{\partial\vec r}{\partial u} = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}, \qquad \frac{\partial\vec r}{\partial v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$$\nabla u = (2,0), \qquad \nabla v = (-2,1).$$

Remark: ∇u is orthogonal to $\frac{\partial\vec r}{\partial v}$ and ∇v is orthogonal to $\frac{\partial\vec r}{\partial u}$, although the elements of each basis are not orthogonal to each other.



3. *(28 pts.)* Find a quadrature rule (approximate integration formula) of the form

$$\int_0^\infty f(t)\,e^{-t}\,dt \approx a_1 f(0) + a_2 f(1) + a_3 f(10)$$

by requiring that the rule gives the exact answer for all f in $\mathcal{P}_2$ (the quadratic polynomials). **Use Cramer's rule to solve for the coefficients, showing intermediate steps.** *Useful information:* $\int_0^\infty t^n e^{-t}\,dt = n!$.

Requiring that the rule gives the right answer on the standard basis for $\mathcal{P}_2$ yields three equations,

$$\begin{aligned} a_1 + a_2 + \quad a_3 &= \int_0^\infty e^{-t}\,dt \quad = 1, \\ a_2 + \ 10a_3 &= \int_0^\infty te^{-t}\,dt \ = 1, \\ a_2 + 100a_3 &= \int_0^\infty t^2e^{-t}\,dt = 2. \end{aligned}$$

The determinant of the system is

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 10 \\ 0 & 1 & 100 \end{vmatrix} = \begin{vmatrix} 1 & 10 \\ 1 & 100 \end{vmatrix} = 90.$$

Therefore,

$$\begin{aligned} a_1 &= \frac{1}{90}\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 10 \\ 2 & 1 & 100 \end{vmatrix} = \frac{1}{90}\left[\begin{vmatrix} 1 & 10 \\ 1 & 100 \end{vmatrix} - \begin{vmatrix} 1 & 10 \\ 2 & 100 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}\right] \\ &= \frac{1}{90}(90 - 80 - 1) = \frac{9}{90} = \frac{1}{10}, \end{aligned}$$

$$a_2 = \frac{1}{90}\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 10 \\ 0 & 2 & 100 \end{vmatrix} = \frac{1}{90}\begin{vmatrix} 1 & 10 \\ 2 & 100 \end{vmatrix} = \frac{80}{90} = \frac{8}{9},$$

$$a_3 = \frac{1}{90}\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} = \frac{1}{90}\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = \frac{1}{90}.$$

Thus, finally,

$$\int_0^\infty f(t)\,e^{-t}\,dt \approx \frac{1}{10}\,f(0) + \frac{8}{9}\,f(1) + \frac{1}{90}\,f(10).$$

4. *(22 pts.)* Let $\vec{B}(x,y,z) = \dfrac{x}{(x^2+y^2)^n}\hat{\imath} + \dfrac{y}{(x^2+y^2)^n}\hat{\jmath}$, where n is an arbitrary, fixed number. (Note that $B_z = 0$.)

(a) Calculate $\iint_S \vec{B}\cdot d\vec{S}$ when S is the piece of cylindrical surface defined in standard cylindrical coordinates by

$$r = 2, \qquad 0 < \theta < \pi, \qquad 0 < z < 3.$$

(The result will be a function of n.)

Easy way: Note that $\vec{B} = \frac{x}{r^{2n}}\hat{\imath} + \frac{y}{r^{2n}}\hat{\jmath}$ is perpendicular to the surface. The unit normal vector is $\hat{n} = \frac{x}{r}\hat{\imath} + \frac{y}{r}\hat{\jmath}$. Thus

$$\vec{B} \cdot \hat{n} = \frac{x^2 + y^2}{r^{2n+1}} = \frac{1}{r^{2n-1}},$$

which is *constant* on the cylinder. So we merely need to multiply by the area of S:

$$\iint_S \vec{B} \cdot d\vec{S} = \frac{1}{2^{2n-1}} \times 6\pi = \frac{3\pi}{4^{n-1}}.$$

Hard way: Since $x = 2\cos\theta$, $y = 2\sin\theta$, $z = z$, we have

$$\begin{aligned}
\iint_S \vec{B} \cdot d\vec{S} &= \iint [B_x\, dy\, dz + B_y\, dz\, dx + B_z\, dx\, dy] \\
&= \iint \left[\frac{2\cos\theta}{4^n}(2\cos\theta d\theta)\, dz + \frac{2\sin\theta}{4^n}\, dz\, (-2\sin\theta)\, d\theta\right] \\
&= \frac{1}{4^{n-1}} \iint \left[\cos^2\theta\, d\theta\, dz + \sin^2\theta\, d\theta\, dz\right] \\
&= \frac{1}{4^{n-1}} \int_0^\pi d\theta \int_0^3 dz = \frac{3\pi}{4^{n-1}}.
\end{aligned}$$

(b) For what value(s) of n does there exist a vector potential $\vec{A}(x, y, z)$ such that $\vec{B} = \nabla \times \vec{A}$ (everywhere except possibly on the axis, $x = y = 0$)?

We need

$$\begin{aligned}
0 = \nabla \cdot \vec{B} &= \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2)^n}\right) + \frac{\partial}{\partial y}\left(\frac{y}{(x^2+y^2)^n}\right) \\
&= \frac{1}{(x^2+y^2)^n} - \frac{2nx^2}{(x^2+y^2)^{n+1}} + \frac{1}{(x^2+y^2)^n} - \frac{2ny^2}{(x^2+y^2)^{n+1}} \\
&= \frac{2x^2 + 2y^2 - 2n(x^2+y^2)}{(x^2+y^2)^{n+1}} \\
&= \frac{2(1-n)}{(x^2+y^2)^n}.
\end{aligned}$$

(This calculation would be easier if we knew the formula for the divergence operator in cylindrical coordinates, but we haven't studied that.) So the needed condition is

$$n = 1.$$

(This means that $\vec{B} = \frac{\hat{n}}{r}$. On the axis, $\vec{B}$, $\hat{n}$, and $\nabla \cdot \vec{B}$ are all undefined. Since $\hat{n}$ points outward, it is geometrically obvious that $\nabla \cdot \vec{B}$ should be regarded as infinite on the axis, much as the divergence of the electric field of a point charge is infinite at the origin. In this case we have "magnetic monopole charge" concentrated along the whole axis.)