Math. 312 20 October 2000

Test B – Solutions

1. *(50 pts.)* Consider Laplace's equation in a "quadrant",

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (0 < x < \infty, \quad 0 < y < \infty),$$

with the boundary conditions

$$u(0, y) = 0, \qquad u(x, 0) = f(x).$$

(a) Solve the equation by separation of variables (or an equivalent transform technique). Assume that $\int_0^\infty |f(x)|^2\, dx < \infty$ and that $|u(x, y)|$ is bounded as $x \to +\infty$ or $y \to +\infty$. (You may skip quickly through the early steps of separation of variables if you're sure your starting point is correct.)

Separation method: Try $u_{\text{sub}} = X(x)Y(y)$. Then

$$+\frac{X''}{X} = -\frac{Y''}{Y} = -\omega^2, \qquad X(0) = 0.$$

Thus $X(x) = \sin(\omega x)$ with no restriction on ω except that it be positive. The solution of the Y equation that vanishes at $+\infty$ is $Y(y) = e^{-\omega y}$. The general solution of the PDE is a superposition

$$\boxed{u(x, y) = \int_0^\infty b(\omega) \sin(\omega x) e^{-\omega y}\, d\omega.}$$

The nonhomogeneous boundary condition is

$$f(x) = \int_0^\infty b(\omega) \sin(\omega x)\, d\omega,$$

and the inverse of this sine transform is

$$\boxed{b(\omega) = \frac{2}{\pi} \int_0^\infty f(x) \sin(\omega x)\, dx.}$$

Transform method: The homogeneous Dirichlet boundary condition and the semi-infinite x interval strongly suggest that we should make a Fourier sine transform with respect to x. Let's (for purposes of this problem) use capital letters for sine transforms of functions denoted by lower-case letters. Thus

$$\boxed{F(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(\omega x)\, dx,} \qquad f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(\omega) \sin(\omega x)\, d\omega,$$

and

$$U(\omega,y)=\sqrt{\frac{2}{\pi}}\int_0^\infty u(x,y)\sin(\omega x)\,dx, \qquad \boxed{u(x,y)=\sqrt{\frac{2}{\pi}}\int_0^\infty U(\omega,y)\sin(\omega x)\,d\omega.}$$

The PDE and BC become

$$-\omega^2 U+\frac{\partial^2 U}{\partial y^2}=0, \qquad U(\omega,0)=F(\omega).$$

The solution for U has the general form $U(\omega,y)=C(\omega)e^{\omega y}+D(\omega)e^{-\omega y}$; the condition of boundedness forces $C=0$; the initial condition forces $D=F$. Thus

$$\boxed{U(\omega,y)=F(\omega)e^{-\omega y}.}$$

Together with the boxed transform formulas for F and u above, this reproduces the solution found by the other method.

(b) Express the solution in terms of a Green function. (You may leave the formula for the Green function as an unevaluated integral. 10 points extra credit if you do evaluate it.)

Combining the boxed formulas from either solution to (a), we get

$$u(x,y)=\frac{2}{\pi}\int_0^\infty\int_0^\infty \sin(\omega x)\sin(\omega z)e^{-\omega y}f(z)\,dz\,d\omega.$$

Writing the ω integration on the inside, we put this into the form

$$u(x,y)=\int_0^\infty G(y,x,z)f(z)\,dz$$

with

$$\boxed{G(y,x,z)=\frac{2}{\pi}\int_0^\infty \sin(\omega x)\sin(\omega z)e^{-\omega y}\,d\omega.}$$

Actually, the integral defining G can be evaluated:

$$G(y,x,z)=\frac{1}{\pi}\left[\frac{y}{y^2+(x-z)^2}-\frac{y}{y^2+(x+z)^2}\right].$$

The easiest (I didn't say "most obvious") way to see this is the "method of images": The solution for u is the restriction to $x>0$ of the solution of Laplace's equation in the entire upper half-plane with boundary data equal to the odd extension of $f(x)$. Thus the Green function is the difference of the Green function for the half-plane with source at z and the one with source at $-z$.

2. *(20 pts.)* Solve

$$\frac{d^2u}{dx^2} = \delta(x-5), \qquad u(0) = 1, \quad \frac{du}{dx}(0) = 0.$$

For $0 < x < 5$ we must have $u = Ax + B$, and the initial conditions say that $A = 0$, $B = 1$. Thus $u(x) = 1$ and $u'(x) = 0$ throughout this interval (and also for $x < 0$). Now for $x > 5$ we have $u = Cx + D$. Continuity requires $u(5^+) = u(5^-) = 1$, so $5C + D = 1$. The remaining condition is $u'(5^+) = u'(5^-) + 1$ (obtained by integrating the differential equation across the singular point). Thus $C = 0 + 1$. Therefore, $D = 1 - 5 = -4$. So we have

$$u(x) = \begin{cases} 1 & \text{if } x \le 5, \\ x - 4 & \text{if } x \ge 5. \end{cases}$$

This can also be written

$$u(x) = 1 + (x-5)h(x-5),$$

where h is the unit step function (Heaviside function).

Alternative method: This problem isn't really a differential equation, just an indefinite integral. Integrating once, we get

$$u'(x) = h(x-5) + C_1\,,$$

and the initial condition $u'(0) = 0$ tells us that $C_1 = 0$. Integrate again:

$$u(x) = (x-5)h(x-5) + C_2\,.$$

[Check: $\frac{d}{dx}[(x-5)h(x-5)] = h(x-5) + (x-5)\delta(x-5)] = h(x-5)$.] Now the condition $u(0) = 1$ implies that $C_2 = 1$.

3. *(30 pts.)* Consider the heat equation on an interval,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < 1, \quad 0 < t < \infty).$$

(a) Find the **steady-state** solution with the boundary data

$$u(t,0) = 2, \qquad u(t,1) = 5.$$

If u does not depend on t, then $\frac{d^2u}{dx^2} = 0$, so

$$u(t,x) = U(x) = Ax + B.$$

Match the boundary conditions: $2 = B$, $5 = A + B$. Thus

$$U(x) = 3x + 2.$$

(b) Show how you would use the solution to (a) to solve the **initial-value** problem with $u(0,x) = f(x)$ (and the same PDE and BC as above). (You won't have time to actually solve the problem; just "set it up".)

Define $w(t,x) = u(t,x) - U(x)$. Then w satisfies

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2}, \qquad w(t,0) = 0, \quad w(t,1) = 0, \qquad w(0,x) = f(x) - U(x).$$

We know how to solve this problem by separation of variables; the coefficients in the solution as a sum of normal modes will be the coefficients in the Fourier sine series of $f - U$.

(c) Discuss what would go wrong, and why, if the boundary conditions in (a) were replaced by

$$\frac{\partial u}{\partial x}(t,0) = 2, \qquad \frac{\partial u}{\partial x}(t,1) = 5.$$

From the calculations in (a) we see that U should be of the form $Ax + B$, but when we impose the boundary conditions we get

$$2 = A = 5,$$

a contradiction. Therefore, no steady-state solution exists. The physical reason for this is that more heat is flowing out at $x = 1$ than is flowing in at $x = 0$, so the rod (one-dimensional body) must cool off instead of staying in a steady state. Mathematically, this is a one-dimensional analogue of Gauss's theorem: Integrating the steady-state differential equation $u''(x) = 0$, we find that $u'(1) - u'(0) = 0$; if the Neumann data at the two ends are not equal, the problem is inconsistent.