Math 412
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Green's Function

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We are given a second order non-homogenous ODE with homogenous boundary conditions and ask for the Green's function. In order to do this, we make use of several identities of Green's functions and the Dirac Delta "function".

$$\frac{d^2y}{dx^2} + 4y = f(x) \tag{1}$$

$$y(0) = 0 \qquad\qquad y(L) = 0$$

The relevant equation for the Green's function is below. Our justification for being able to write such an odd looking equation is given later in this document. Notice we solve the equation in two regions, each corresponding to either side of the Dirac Delta impulse.

$$\frac{d^2G}{dx^2} + 4G = \delta(x - z) \tag{2}$$

$$G(x,z)_< = A\cos(2x) + B\sin(2x) \qquad (x < z) \tag{3}$$

$$G(x,z)_> = C\cos(2(x - L)) + D\sin(2(x - L)) \qquad (x > z) \tag{4}$$

When $x < z$, the boundary condition implies that $A = 0$. When $x > z$, the boundary condition implies that $C = 0$.

We can evaluate the integral of Eqn. (5).

$$\int_0^L (\frac{d^2G}{dx^2} + 4G)dz = \int_0^L \delta(x - z)dz \tag{5}$$

$$\frac{dG}{dx}\mid_{z^+} - \frac{dG}{dx}\mid_{z^-} = 1 \tag{6}$$

The Green's function is continuous in the neighborhood of z, and in fact everywhere on the interval. If it weren't, then $\frac{d^2G}{dx^2}$ would be worse-behaved than $\delta(x-z)$, which is a clear sign that G isn't such. The reason the G term itself does not contribute to the integral is that, since $\delta(x-z)$ only contributes in the interval $[z+\epsilon, z-\epsilon]$ where ϵ is an arbitrarily small number such that $\epsilon \in \mathbf{R}$, we can reduce the integration interval to such. On this interval, by our stipulation that G is continuous, G will not vary since ϵ is infinitely small.

The continuity condition and the condition we just derived result in a linear system of order 2. We write this in matrix form, and then seek the determinant:

$$\begin{pmatrix} \sin(2z) & -\sin(2(z-L)) \\ \cos(2z) & -\cos(2(z-L)) \end{pmatrix} \begin{pmatrix} B \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} \tag{7}$$

After computing the determinant and simplifying with a trigonometric identity, the result is

$$\Delta = -\sin(2L) \tag{8}$$

Using Cramer's rule, we have the following

$$B = \frac{1}{\Delta} \begin{pmatrix} 0 & -\sin(2(z-L)) \\ \frac{1}{2} & -\cos(2(z-L)) \end{pmatrix} = \frac{-\sin(2(z-L))}{2\sin(2L)} \tag{9}$$

$$D = \frac{1}{\Delta} \begin{pmatrix} \sin(2z) & 0 \\ \cos(2z) & \frac{1}{2} \end{pmatrix} = \frac{-\sin(2z)}{2\sin(2L)} \tag{10}$$

Therefore, our final Green's function is

$$G(x,z)_< = \frac{-\sin(2(z-L))\sin(2x)}{\sin(2L)} \qquad (x < z) \tag{11}$$

$$G(x,z)_> = \frac{-\sin(2z)\sin(2(x-L))}{\sin(2L)} \qquad (x > z) \tag{12}$$

However, it is clear that for some values of L, $\Delta = 0$ may vanish. In this case, our Green's function diverges. This becomes clear upon considering the original ODE's homogenous equation, which when solved with the boundary data, results in the series $\omega_n^2 = \frac{n\pi}{2}$. It's no coincidence that setting $\Delta = 0$ results in the same series. Then, the solution to the ODE is non-unique, and in our case there are infinitely many solutions. No single Green's function could hope to represent all this.