The Wave Equation

This introductory example will have three parts.*

1. I will show how a particular, simple partial differential equation (PDE) arises in a physical problem.

2. We'll look at its solutions, which happen to be unusually easy to find in this case.

3. We'll solve the equation again by *separation of variables*, the central theme of this course, and see how *Fourier series* arise.

The *wave equation* in two variables (one space, one time) is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \,,$$

where c is a constant, which turns out to be the speed of the waves described by the equation.

Most textbooks derive the wave equation for a vibrating string (e.g., Haberman, Chap. 4). It arises in many other contexts — for example, light waves (the electromagnetic field). For variety, I shall look at the case of sound waves (motion in a gas).

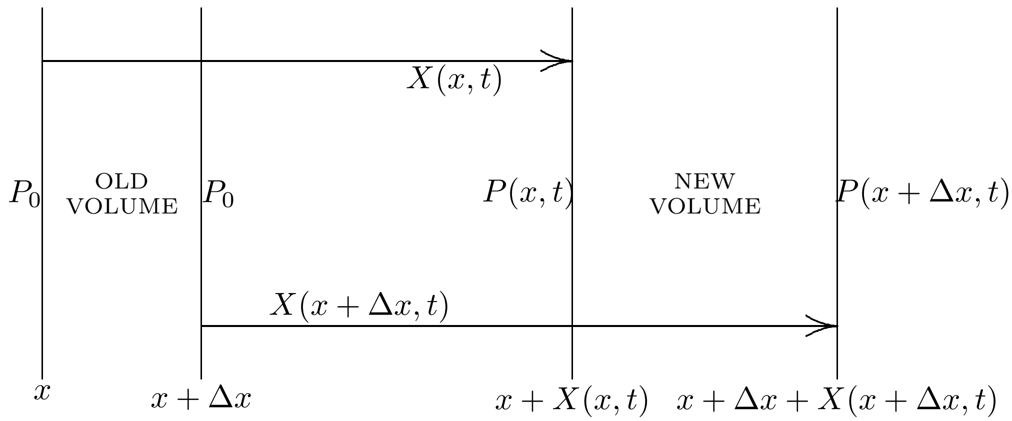
Sound waves

Reference: *Feynman Lectures in Physics*, Vol. 1, Chap. 47.

We assume that the gas moves back and forth in one dimension only (the x direction). If there is no sound, then each bit of gas is at rest at some place (x, y, z). There is a uniform equilibrium density ρ_0 (mass per unit volume) and pressure P_0 (force per unit area). Now suppose the gas moves; all gas in the layer at x moves the same distance, $X(x)$, but gas in other layers move by different distances. More precisely, at each time t the layer originally at x is displaced to $x + X(x,t)$. There it experiences a new density and pressure, called

$$\rho = \rho_0 + \rho_1(x,t), \qquad P = P_0 + P_1(x,t).$$

* Simultaneously, students should be reading about another introductory example, the *heat equation*, in Chapters 1 and 2 of Haberman's book. (See also Appendix A of these notes.)



Given this scenario, Newton's laws imply a PDE governing the motion of the gas. The input to the argument is three physical principles, which will be translated into three equations that will imply the wave equation.

I. **The motion of the gas changes the density.** Take a slab of thickness Δx in the gas at rest. The total amount of gas in the slab (measured by mass) is

$$\rho_0 \times \text{volume} = \rho_0\, \Delta x \times \text{area}.$$

We can consider a patch with area equal to 1. In the *moving* gas at time t, this *same* gas finds itself in a new volume (area times thickness)

$$(\text{area} \times\ )\,\{[x + \Delta x + X(x + \Delta x, t)] - [x + X(x, t)]\} \equiv \Delta x_{\text{new}}\,.$$

(Cancel x.) Thus $\rho_0 \Delta x = \rho \Delta x_{\text{new}}$. If Δx is small, we have

$$X(x + \Delta x, t) - X(x, t) \approx \frac{\partial X}{\partial x} \cdot \Delta x;$$

$$\rho_0 \Delta x = \rho \left(\Delta x + \frac{\partial X}{\partial x}\, \Delta x \right).$$

(Cancel Δx.) So

$$\rho_0 = (\rho_0 + \rho_1) \frac{\partial X}{\partial x} + \rho_0 + \rho_1\,.$$

Since $\rho_1 \ll \rho_0$, we can replace $\rho_0 + \rho_1$ by ρ_0 in its first occurrence — but not the second, where the ρ_0 is cancelled, leaving ρ_1 as the most important term. Therefore, we have arrived (essentially by geometry) at

$$\rho_1 = -\rho_0 \frac{\partial X}{\partial x}\,. \tag{I}$$

II. **The change in density corresponds to a change in pressure.** (If you push on a gas, it pushes back, as we know from feeling balloons.) Therefore, $P = f(\rho)$, where f is some *increasing* function.

$$P_0 + P_1 = f(\rho_0 + \rho_1) \approx f(\rho_0) + \rho_1 f'(\rho_0)$$

since ρ_1 is small. (Cancel P_0.) Now $f'(\rho_0)$ is greater than 0; call it c^2:

$$P_1 = c^2 \rho_1 \,. \qquad \text{(II)}$$

III. **Pressure inequalities generate gas motion.** The force on our slab (measured positive to the right) equals the pressure acting on the left side of the slab minus the pressure acting on the right side (times the area, which we set to 1). But this force is equal to mass times acceleration, or

$$(\rho_0 \Delta x) \frac{\partial^2 X}{\partial t^2} \,.$$

$$\rho_0 \Delta x \,\frac{\partial^2 X}{\partial t^2} = P(x,t) - P(x+\Delta x, t) \approx -\,\frac{\partial P}{\partial x}\,\Delta x.$$

(Cancel Δx.) But $\partial P_0 / \partial x = 0$. So

$$\rho_0 \,\frac{\partial^2 X}{\partial t^2} = -\,\frac{\partial P_1}{\partial x} \,. \qquad \text{(III)}$$

Now put the three equations together. Substituting (I) into (II) yields

$$P_1 = -c^2 \rho_0 \,\frac{\partial X}{\partial x} \,.$$

Put that into (III):

$$\rho_0 \,\frac{\partial^2 X}{\partial t^2} = +c^2 \rho_0 \,\frac{\partial^2 X}{\partial x^2} \,.$$

Finally, cancel ρ_0:

$$\boxed{\frac{\partial^2 X}{\partial t^2} = c^2 \,\frac{\partial^2 X}{\partial x^2} \,.}$$

Remark: The thrust of this calculation has been to eliminate all variables but one. We chose to keep X, but could have chosen P_1 instead, getting

$$\frac{\partial^2 P_1}{\partial t^2} = c^2 \,\frac{\partial^2 P_1}{\partial x^2} \,.$$

(Note that P_1 is proportional to $\partial X / \partial x$ by (II) and (I).) Also, the same equation is satisfied by the gas *velocity*, $v(x,t) \equiv \partial X / \partial t$.

The wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}\,,$$

can be solved by a special trick. (The rest of this course is devoted to other PDEs for which this trick does not work!)

Make a change of independent variables:

$$w \equiv x + ct, \qquad z \equiv x - ct.$$

The dependent variable u is now regarded as a function of w and z. To be more precise one could write $u(x,t) = \tilde{u}(w,z)$ (but I won't). We are dealing with a *different* function but the *same* physical quantity.

By the chain rule, acting upon any function we have

$$\frac{\partial}{\partial t} = \frac{\partial w}{\partial t}\frac{\partial}{\partial w} + \frac{\partial z}{\partial t}\frac{\partial}{\partial z} = c\,\frac{\partial}{\partial w} - c\,\frac{\partial}{\partial z}\,,$$

$$\frac{\partial}{\partial x} = \frac{\partial w}{\partial x}\frac{\partial}{\partial w} + \frac{\partial z}{\partial x}\frac{\partial}{\partial z} = \frac{\partial}{\partial w} + \frac{\partial}{\partial z}\,.$$

Therefore,

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= c\left(\frac{\partial}{\partial w} - \frac{\partial}{\partial z}\right)\left[c\left(\frac{\partial}{\partial w} - \frac{\partial}{\partial z}\right)u\right] \\ &= c^2\left(\frac{\partial^2 u}{\partial w^2} - 2\,\frac{\partial^2 u}{\partial w\,\partial z} + \frac{\partial^2 u}{\partial z^2}\right).\end{aligned}$$

Similarly,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial w^2} + 2\,\frac{\partial^2 u}{\partial w\,\partial z} + \frac{\partial^2 u}{\partial z^2}\,.$$

Thus the wave equation is

$$0 = \frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\,\frac{\partial^2 u}{\partial t^2}\right) = \frac{\partial^2 u}{\partial w\,\partial z}\,.$$

This new equation is easily solved. We can write it in the form

$$\frac{\partial}{\partial w}\left(\frac{\partial u}{\partial z}\right) = 0.$$

Then it just says that $\dfrac{\partial u}{\partial z}$ is a constant, as far as w is concerned. That is,

$$\frac{\partial u}{\partial z} = \gamma(z) \quad \text{(a function of } z \text{ only).}$$

Consequently,

$$u(w,z) = \int_{z_0}^{z} \gamma(\tilde{z})\, d\tilde{z} + C(w),$$

where z_0 is some arbitrary starting point for the indefinite integral. Note that the constant of integration will in general depend on w. Now since γ was arbitrary, its indefinite integral is an essentially arbitrary function too, and we can forget γ and just call the first term $B(z)$:

$$u(w,z) = B(z) + C(w).$$

(The form of the result is symmetrical in z and w, as it must be, since we could equally well have worked with the equation in the form $\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial w}\right) = 0$.)

So, we have found the *general solution* of the wave equation to be

$$\boxed{u(x,t) = B(x - ct) + C(x + ct),}$$

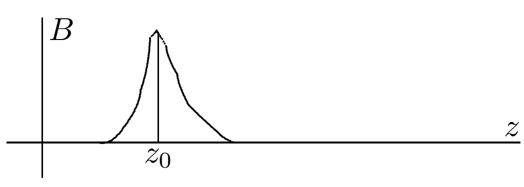
where B and C are arbitrary functions. (Technically speaking, we should require that the second derivatives B'' and C'' exist and are continuous, to make all our calculus to this point legal. However, it turns out that the d'Alembert formula remains meaningful and correct for choices of B and C that are much rougher than that.)

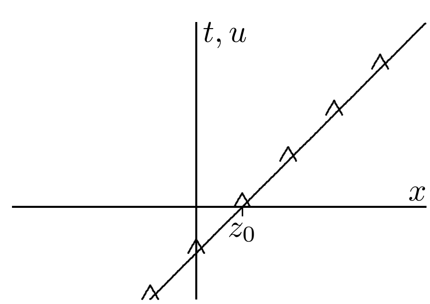
Interpretation

What sort of function is $B(x - ct)$? It is easiest to visualize if $B(z)$ has a peak around some point $z = z_0$. Contemplate $B(x - ct)$ as a function of x for a fixed t: It will have a peak in the neighborhood of a point x_0 satisfying $x_0 - ct = z_0$, or

$$x_0 = z_0 + ct.$$

That is, the "bump" moves to the right with velocity c, keeping its shape exactly.





(Note that in the second drawing we have to plot u on the same axis as t. Such pictures should be thought of as something like a strip of movie film which we are forced to look at without the help of a projector.)*

Similarly, the term $C(x+ct)$ represents a wave pattern which moves rigidly to the *left* at the wave velocity $-c$. If both terms are present, and the functions are sharply peaked, we will see the two bumps collide and pass through each other. If the functions are not sharply peaked, the decomposition into left-moving and right-moving parts will not be so obvious to the eye.

Initial conditions

In a concrete problem we are interested not in the most general solution of the PDE but in the particular solution that solves the problem! How much additional information must we specify to fix a unique solution? The *two arbitrary functions* in the general solution recalls the *two arbitrary constants* in the general solution of a second-order *ordinary* differential equation (ODE), such as

$$\frac{d^2u}{dt^2} + 4u = 0; \qquad u(t) = B\sin(2t) + A\cos(2t).$$

In that case we know that the two constants can be related to two *initial conditions* (IC):

$$u(0) = A, \qquad \frac{du}{dt}(0) = 2B.$$

Similarly, for the wave equation the two functions $B(z)$ and $C(w)$ can be related to initial data measured at, say, $t=0$. (However, things will not be so simple for other second-order PDEs.)

Let's assume for the moment that our wave equation applies for *all* values of x and t:

$$-\infty < x < \infty, \qquad -\infty < t < \infty.$$

We consider initial data at $t=0$:

$$u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x).$$

The d'Alembert solution implies

$$f(x) = B(x) + C(x), \qquad g(x) = -cB'(x) + cC'(x).$$

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* In advanced physics, especially relativistic physics, it is standard to plot t on the *vertical* axis and x on the horizontal, even though for particle motion t is the independent variable and x the dependent one.

The second condition implies

$$-B(x) + C(x) = \int \frac{g(x)}{c}\,dx = G(x) + A,$$

where G is any antiderivative of g/c, and A is an unknown constant of integration. Solve these equations for B and C:

$$B(x) = \tfrac{1}{2}[f(x) - G(x) - A], \qquad C(x) = \tfrac{1}{2}[f(x) + G(x) + A].$$

We note that A cancels out of the total solution, $B(x-ct) + C(x+ct)$. (Being constant, it qualifies as both left-moving and right-moving; so to this extent, the decomposition of the solution into left and right parts is ambiguous.) So we can set $A = 0$ without losing any solutions. Now our expression for the solution in terms of the initial data is

$$\boxed{u(x,t) = \tfrac{1}{2}[f(x+ct) + f(x-ct)] + \tfrac{1}{2}[G(x+ct) - G(x-ct)].}$$

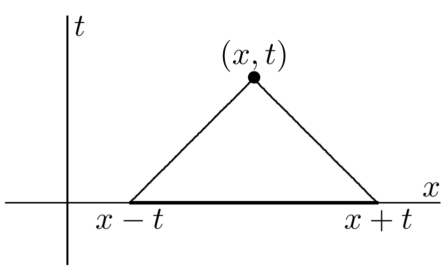
This is the first form of d'Alembert's fundamental formula. To get the second form, use the fundamental theorem of calculus to rewrite the G term as an integral over g:

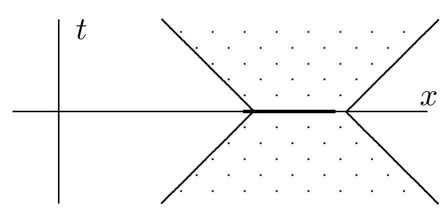
$$\boxed{u(x,t) = \tfrac{1}{2}[f(x+ct) + f(x-ct)] + \tfrac{1}{2c}\int_{x-ct}^{x+ct} g(w)\,dw.}$$

This formula demonstrates that the value of u at a point (x,t) depends only on the part of the initial data representing "stuff" that has had time to reach x while traveling at speed c — that is, the data $f(w,0)$ and $g(w,0)$ on the *interval of dependence*

$$x - ct < w < x + ct \qquad (\text{for } t > 0).$$

Conversely, any interval on the initial data "surface" (the line $t = 0$, in the two-dimensional case) has an expanding *region of influence* in space-time, beyond which its initial data are irrelevant. In other words, "signals" or "information" are carried by the waves with a finite maximum speed. These properties continue to hold for other wave equations (for example, in higher-dimensional space), even though in those cases the simple d'Alembert formula for the solution is lost and the waves no longer keep exactly the same shape as they travel.





In realistic problems one is usually concerned with only part of space (e.g, sound waves in a room). What happens to the waves at the edge of the region affects what happens inside. We need to specify this boundary behavior, in addition to initial data, to get a unique solution. To return to our physical example, if the sound waves are occurring in a closed pipe (of length L), then the gas should be motionless at the ends:

$$X(0,t) = 0 = X(L,t).$$

Mathematically, these are called *Dirichlet boundary conditions* (BC). In contrast, if the pipe is open at one end, then to a good approximation the pressure at that point will be equal to the outside pressure, P_0 . By our previous remark, this implies that the *derivative* of X vanishes at that end; for instance,

$$\frac{\partial X}{\partial x}(0,t) = 0$$

instead of one of the previous equations. This is called a *Neumann boundary condition.*

When a wave hits a boundary, it *reflects*, or "bounces off". Let's see this mathematically. Consider the interval $0 < x < \infty$ and the Dirichlet condition

$$u(0,t) = 0.$$

Of course, we will have initial data, f and g, defined for $x \in (0,\infty)$.

We know that

$$u(x,t) = B(x-ct) + C(x+ct) \qquad (1)$$

and

$$B(w) = \tfrac{1}{2}[f(w) - G(w)], \qquad C(w) = \tfrac{1}{2}[f(w) + G(w)], \qquad (2)$$

where f and $cG' \equiv g$ are the initial data. However, if we try to calculate u from (1) for $t > x/c$, we find that (1) directs us to evaluate $B(w)$ for *negative* w; this is not defined in our present problem! To see what is happening, start at (x,t) and trace a right-moving ray backwards in time: It will run into the wall (the positive t-axis), not the initial-data surface (the positive x-axis).

Salvation is at hand through the boundary condition, which gives us the additional information

$$B(-ct) = -C(ct). \qquad (3)$$

For $t > 0$ this condition determines B(negative argument) in terms of C(positive argument). For $t < 0$ it determines C(negative argument) in terms of B(positive argument). Thus B and C are uniquely determined for all arguments by (2) and (3) together.

In fact, there is a convenient way to represent the solution $u(x,t)$ in terms of the initial data, f and g. Let us *define* $f(x)$ and $g(x)$ for negative x by requiring (2) to hold for negative values of w as well as positive. If we let $y \equiv ct$, (2) and (3) give (for all y)

$$f(-y) - G(-y) = -f(y) - G(y). \tag{4}$$

We would like to solve this for $f(-y)$ and $G(-y)$, assuming y positive. But for that we need an independent equation (to get two equations in two unknowns). This is provided by (4) with negative y; write $y = -x$ and interchange the roles of right and left sides:

$$f(-x) + G(-x) = -f(x) + G(x). \tag{5}$$

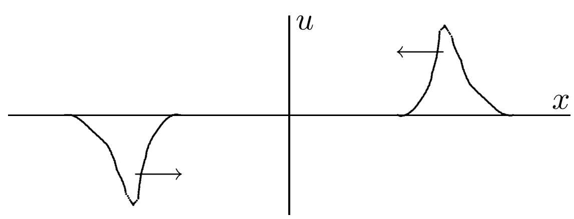
Rewrite (4) with $y = +x$ and solve (4) and (5): For $x > 0$,

$$f(-x) = -f(x), \qquad G(-x) = G(x). \tag{6}$$

What we have done here is to define *extensions* of f and g from their original domain, $x > 0$, to the whole real line. The conditions (6) define the *odd extension* of f and the *even extension* of G. (It's easy to see that $g = cG'$ is then odd, like f.) We can now solve the wave equation in all of $\mathbf{R}^2$ ($-\infty < x < \infty$, $-\infty < t < \infty$) with these odd functions f and g as initial data. The solution is given by d'Alembert's formula,

$$u(x,t) = \tfrac{1}{2}[f(x+ct) + f(x-ct)] + \tfrac{1}{2}[G(x+ct) - G(x-ct)],$$

and it is easy to see that the boundary condition, $u(0,t) = 0$, is satisfied, because of the parity (evenness and oddness) of the data functions. Only the part of the solution in the region $x > 0$ is physical; the other region is fictitious. In the latter region we have a "ghost" wave which is an *inverted mirror image* of the physical solution.



The calculation for Neumann conditions goes in very much the same way, leading to *even* extensions of f and g. The result is that the pulse reflects *without* turning upside down. Approximations to the "ideal" Dirichlet and Neumann boundary conditions are provided by a standard high-school physics experiment with Slinky™ springs. A small, light spring and a large, heavy one are attached end to end. When a wave traveling along the light spring hits the junction, the heavy spring remains almost motionless and the pulse reflects inverted. When the wave is in the heavy spring, the light spring serves merely to stabilize the apparatus; it carries off very little energy and barely constrains the motion of the end of the heavy spring. The pulse, therefore, reflects without inverting.

Suppose that the spatial domain is $0 < x < L$ with a Dirichlet condition at each end. The condition $u(0,t) = 0$ can be treated by constructing odd and even extensions as before. The condition $u(L,t) = 0$ implies, for all t,

$$\begin{aligned} 0 &= B(L-ct) + C(L+ct) \\ &= \tfrac{1}{2}[f(L-ct) - G(L-ct)] + \tfrac{1}{2}[f(L+ct) + G(L+ct)]. \end{aligned} \tag{7}$$

Treating this equation as we did (4), we find an extension of f and G beyond the right end of the interval:

$$\begin{aligned} f(L+ct) &= -f(L-ct) = +f(-L+ct), \\ G(L+ct) &= G(L-ct) = G(-L+ct). \end{aligned}$$

(In more detail: Treat $f(L+ct)$ and $G(L+ct)$ with $t > 0$ as the unknowns. Replacing t by $-t$ in (7) gives two independent equations to be solved for them.) Finally, set $ct = s + L$:

$$f(s+2L) = f(s), \qquad G(s+2L) = G(s) \tag{8}$$

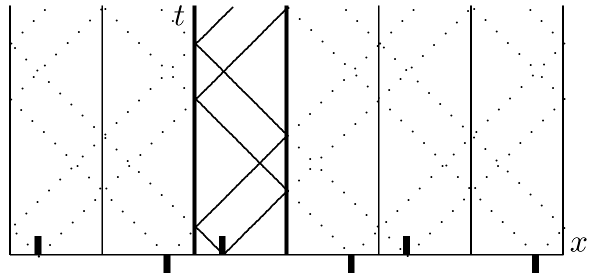
for all s. That is, the properly extended f and G (or g) are *periodic* with period $2L$.

Here is another way to derive (8): Let's go back to the old problem with just one boundary, and suppose that it sits at $x = L$ instead of $x = 0$. The basic geometrical conclusion can't depend on where we put the zero of the coordinate system: It must still be true that the extended data function is the odd (i.e., inverted) reflection of the original data through the boundary. That is, the value of the function at the point at a distance s to the left of L is minus its value at the point at distance s to the right of L. If the coordinate of the first point is x, then (in the case $L > 0$) s equals $L - x$, and therefore the coordinate of the second point is $L + s = 2L - x$. (This conclusion is worth remembering for future use: *The reflection of the point x through a boundary at L is located at $2L - x$.*) Therefore, the extended data function satisfies

$$f(x) = -f(2L - x).$$

In the problem with two boundaries, it also satisfies $f(x) = -f(-x)$, and thus $f(2L - x) = f(x)$, which is equivalent to the first half of (8) (and the second half can be proved in the same way).

The d'Alembert formula with these periodic initial data functions now gives a solution to the wave equation that satisfies the desired boundary and initial conditions. If the original initial data describe a single "bump", then the extended initial data describe an infinite sequence of image bumps, of alternating sign, as if space were filled with infinitely many parallel mirrors reflecting each other's images. Part of each bump travels off in each direction at speed c. What this really means is that the two wave pulses from the original, physical bump will suffer many reflections from the two boundaries. When a "ghost" bump penetrates into the physical region, it represents the result of one of these reflection events.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