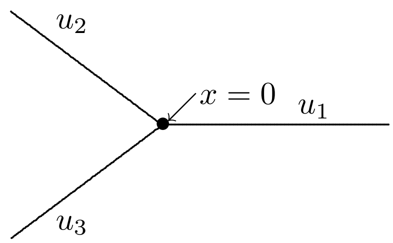
D'Alembert Solution on a Metric Tree Graph
(three infinite strings tied together at a point)



On each string there is a coordinate x ranging from 0 to $+\infty$. The dependent variable on string j is u_j .

PDE:
$$\frac{\partial^2 u_j}{\partial t^2} = \frac{\partial^2 u_j}{\partial x^2} \, .$$

IC:
$$u_j(x,0) = f_j(x) \text{ and } \frac{\partial u_j}{\partial t}(x,0) = g_j(x) \equiv G_j'(x) \text{ for } x > 0.$$

BC1:
$$u_j(0,t) \text{ is the same for all } j \qquad \text{(continuity).}$$

BC2:
$$\sum_{j=1}^{3} \frac{\partial u_j}{\partial x}(0,t) = 0.$$

Suppose that $f_j = 0 = g_j$ for $j \neq 1$. What do you expect to happen to the pulse?

We know that on each string the solution must have the form

$$u_j(x,t) = B_j(x-t) + C_j(x+t) \tag{1}$$

where

$$B_j(x) = \tfrac{1}{2}[f_j(x) - G_j(x)], \quad C_j(x) = \tfrac{1}{2}[f_j(x) + G_j(x)]. \tag{2}$$

This construction completely handles the PDE and IC. Formally applying the boundary conditions, we get

$$B_1(-t) + C_1(t) = B_2(-t) + C_2(t) = B_3(-t) + C_3(t) \tag{3}$$

and

$$B_1'(-t) + C_1'(t) + B_2'(-t) + C_2'(t) + B_3'(-t) + C_3'(t) = 0.$$

The last equation can be integrated as

$$-B_1(-t) + C_1(t) - B_2(-t) + C_2(t) - B_3(-t) + C_3(t) = \tfrac{1}{2}A, \tag{4}$$

where A is an unknown constant that we expect to cancel out of the final solution.

Equations (3) and (4) should be written out in terms of f_j and G_j. As in the standard problem on the half-line, these quantities are not, *a priori*, defined for negative arguments, and we adopt the philosophy that (3) and (4) should be used to define them there. Each of those equations yields two equations, one for positive t and an independent one for negative t. Writing the unknowns on the left and the knowns on the right, we then come up with

$$f_1(-x) - G_1(-x) - f_2(-x) + G_2(-x) = -f_1(x) - G_1(x) + f_2(x) + G_2(x) \qquad (\text{for } t = +x),$$

$$f_1(-x) + G_1(-x) - f_2(-x) - G_2(-x) = -f_1(x) + G_1(x) + f_2(x) - G_2(x) \qquad (\text{for } t = -x),$$

and two identical equations with subscript 2 replaced by 3, and similarly

$$-f_1(-x) - f_2(-x) - f_3(-x) + G_1(-x) + G_2(-x) + G_3(-x) = A - f_1(x) - \cdots - G_1(x) - \cdots,$$

$$f_1(-x) + f_2(-x) + f_3(-x) + G_1(-x) + G_2(-x) + G_3(-x) = A + f_1(x) + \cdots - G_1(x) - \cdots.$$

These are six equations in six unknowns.

However, the f and G problems can be quickly separated. Adding and subtracting each pair of equations, we get

$$f_1(-x) - f_2(-x) = -f_1(x) + f_2(x), \tag{5}$$

$$f_1(-x) - f_3(-x) = -f_1(x) + f_3(x), \tag{6}$$

$$f_1(-x) + f_2(-x) + f_3(-x) = f_1(x) + f_2(x) + f_3(x) \qquad (6)$$

and

$$G_1(-x) - G_2(-x) = G_1(x) - G_2(x),$$

$$G_1(-x) - G_3(-x) = G_1(x) - G_3(x),$$

$$G_1(-x) + G_2(-x) + G_3(-x) = A - G_1(x) - G_2(x) - G_3(x).$$

Now add (5), (6), and (7) and divide by 3 to get

$$f_1(-x) = -\tfrac{1}{3} f_1(x) + \tfrac{2}{3}[f_2(x) + f_3(x)]. \tag{8}$$

We can almost declare victory now! By symmetry it is obvious that

$$f_2(-x) = -\tfrac{1}{3} f_2(x) + \tfrac{2}{3}[f_1(x) + f_3(x)], \tag{9}$$

$$f_3(-x) = -\tfrac{1}{3} f_3(x) + \tfrac{2}{3}[f_1(x) + f_2(x)].$$

Similarly, from the G system we get

$$G_1(-x) = \tfrac{1}{3} A + \tfrac{1}{3} G_1(x) - \tfrac{2}{3}[G_2(x) + G_3(x)], \tag{10}$$

$$G_2(-x) = \tfrac{1}{3} A + \tfrac{1}{3} G_2(x) - \tfrac{2}{3}[G_1(x) + G_3(x)], \tag{11}$$

and so on for G_3.

Remember that we want to use these formulas in (1) and (2), which combine as

$$u_j(x,t) = \tfrac{1}{2}[f_j(x-t) - G_j(x-t)] + \tfrac{1}{2}[f_j(x+t) + G_j(x+t)]. \tag{12}$$

We can see that the As will indeed cancel, and we choose G_j so that $G_j(0) = 0$.

Let us write the full final answer only for the case that

$$f_j(x) = 0 = g_j(x) \text{ for } j \neq 1 \text{ (and } x > 0).$$

Also, we consider only $t > 0$. Then $x + t > 0$, so the second half of (12) never requires the reflected functions (8)–(11); in particular, it is 0 in u_2 and u_3 in our case. (These terms describe the original incoming waves, which have not yet been affected by the boundary conditions.) For the outgoing terms of (12), let's look first at u_2. From (9) and (11) we get

$$u_2(x,t) = \begin{cases} 0 & \text{if } x-t > 0, \\ \tfrac{1}{3}[f_1(t-x) + G_1(t-x)] & \text{if } x-t < 0. \end{cases}$$

Of course, u_3 will be the same. So a certain fraction of the incident pulse in the first string passes through into each of the other two strings. As for u_1, for $x - t > 0$ ("early times") it is just given by (12) with the original unreflected functions. For $x - t < 0$ ("late times") we get from (12), (8), and (10)

$$u_1(x,t) = -\tfrac{1}{6}[f_1(t-x) + G_1(t-x)] + \tfrac{1}{2}[f_1(x+t) + G_1(x+t)].$$

That is, a part of the incident pulse is reflected back into the first string with a change of sign.

Notice that, if we omit the overall factors $\frac{1}{2}$, the amplitude of each of the transmitted waves is $\frac{2}{3}$ and that of the reflected wave is $-\frac{1}{3}$, relative to the amplitude 1 of the incident wave. If there had been N strings instead of three, these numbers would have been $\frac{2}{N}$ and $1 - \frac{2}{N}$. The energy in a wave is proportional to the square of the amplitude, and conservation of energy is shown by the fact that the squares of the outgoing amplitudes add up to that of the incoming one:

$$\frac{4}{9} + \frac{4}{9} + \frac{1}{9} = 1,$$

or, in the general case,

$$(N-1)\frac{4}{N^2} + \left(1 - \frac{2}{N}\right)^2 = 1. \tag{13}$$

Perhaps surprisingly, the more strings there are, the less energy gets through: As $N \to \infty$ the first term in (13) goes to 0 and the second term goes to 1.