Airy's Integral

Solution by Fourier transform

Airy functions are solutions of

$$\frac{d^2y}{dx^2} = xy.$$

It is easy to see that solutions of

$$\frac{d^2y}{dx^2} = \omega^2 xy \qquad \text{and} \qquad \frac{d^2y}{dx^2} = xy - Ey$$

are also Airy functions with rescaled or displaced arguments, and hence the limits of large $|x|$, large ω, and large $|E|$ are essentially the same things. There is a distinguished solution, called $\mathrm{Ai}(x)$, that decays rapidly as $x \to +\infty$. All solutions linearly independent of Ai grow rapidly in that limit, and one of them is conventionally named $\mathrm{Bi}(x)$. As $x \to -\infty$ both Ai and Bi are oscillatory with a slow decay, like Bessel functions.

Airy's is one of the few variable-coefficient ordinary differential equations that can be easily solved by a Fourier transform. Let's define the transform by

$$\hat{y}(p) = \int_{-\infty}^{\infty} e^{-ipx} y(x)\, dx.$$

Then

$$\frac{d}{dx} \mapsto +ip, \qquad \frac{d}{dp} \mapsto -ix.$$

The equation becomes $-p^2\hat{y} = i\frac{d\hat{y}}{dp}$, or

$$\frac{d\hat{y}}{dp} = ip^2\hat{y}.$$

The only solutions are multiples of

$$\hat{y}(p) = e^{ip^3/3}.$$

So we ought to get a solution of Airy's equation from the formula

$$y(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ipx} e^{ip^3/3}\, dp \equiv \mathrm{Ai}(x) \qquad (*)$$

(*Airy's integral*).

But we were supposed to get *two* independent solutions, right? The Fourier transform turned a second-order equation into a first-order one, so we lost half of the solutions (by dimension count) at the first step. The solutions that are not proportional to Ai (i.e., that are $a\,\mathrm{Ai} + b\,\mathrm{Bi}$ with $b \neq 0$) grow so rapidly at infinity that their Fourier transforms are not well-defined. (They can be constructed, however, in terms of integrals in the complex plane with integrands that are essentially the same as in Airy's integral.)

Another initially puzzling thing about Airy's integral is that it may appear to diverge at infinity. Recall that a *series* $\sum_{n=1}^{\infty} e^{i\theta_n}$ with real θ_n must diverge, because the terms do not tend to zero (except in the extended sense in which such a series can converge to a Dirac delta function, say). However, an improper integral can converge even if the integrand maintains modulus 1, just because the speed of oscillation increases as $p \to \pm\infty$. In the case at hand that can be checked by integration by parts to reduce the formula to an integral with a p^3 in the denominator plus other manifestly finite terms.

Oscillatory region: Large negative x

Let $\phi(p) = px + p^3/3$, so that the integrand of $(*)$ is $e^{i\phi}$. We apply the method of *stationary phase*: One's intuition is that the integral should receive very little contribution from intervals of p where the integrand is rapidly oscillating. In other words, most of the integral comes from the vicinity of points where ϕ is not changing at all. We solve

$$0 = \phi'(p_0) = x + {p_0}^2.$$

If $x > 0$ there are no real solutions for p_0 , and we put that case aside. If $x \equiv -z < 0$, then $p_0 = \pm\sqrt{z}$.

Now expand the exponent in a power series around p_0:

$$\begin{aligned}
2\pi\,\mathrm{Ai}(x) &= \int_{-\infty}^{\infty} e^{i\phi(p)}\,dp \\
&\approx \sum_{\pm} \int_{-\infty}^{\infty} e^{i[\phi(p_0)+\frac{1}{2}(p-p_0)^2\phi''(p_0)+\cdots]}\,dp \\
&\approx \sum_{\pm} \int_{-\infty}^{\infty} e^{i[\mp z^{3/2}\pm z^{3/2}/3+\frac{1}{2}(p\mp\sqrt{z})^2(\pm 2\sqrt{z})]}\,dp \\
&= \sum_{\pm} e^{\mp 2iz^{3/2}/3} \int_{-\infty}^{\infty} e^{\pm i\sqrt{z}(p\mp\sqrt{z})^2}\,dp \\
&= \sum_{\pm} e^{\mp 2iz^{3/2}/3} \sqrt{\frac{\pi}{\pm i\sqrt{z}}} \\
&= \sum_{\pm} \sqrt{\pi}(-x)^{-1/4} e^{\pm i\pi/4} e^{\mp 2iz^{3/2}/3} \\
&= 2\sqrt{\pi}(-x)^{-1/4} \cos\left(\frac{2}{3}(-x)^{3/2} - \frac{\pi}{4}\right) .
\end{aligned}$$

This approximation is more accurate the larger z is.

More generally, *any* solution of Airy's equation will behave at large $-x$ as some linear combination of $(-x)^{-1/4}e^{\pm 2i(-x)^{3/2}/3}$. This fact could have established by a calculation similar to the one we carried out for the Bessel function (see also below). The analysis of Airy's integral goes that one one better; it shows *which* linear combination corresponds to the particular solution Ai.

Decay region: Large positive x

Since ϕ' is never zero in this case, we can integrate by parts to our hearts' content:

$$\begin{aligned}
2pi\,\mathrm{Ai}(x) &= \int_{-\infty}^{\infty} \frac{1}{i\phi'(p)}\,\frac{d}{dp} e^{i\phi}\,dp \\
&= -\int_{-\infty}^{\infty} \frac{d}{dp}\left(\frac{1}{i\phi'(p)}\right) e^{i\phi}\,dp \\
&= -i\int_{-\infty}^{\infty} \frac{\phi''}{(\phi')^2}\, e^{i\phi}\,dp \\
&= -2i\int_{-\infty}^{\infty} \frac{p}{(x+p^2)^2}\, e^{i\phi}\,dp = O(x^{-2}) \\
&= -2i\int_{-\infty}^{\infty} \frac{p}{(\phi')^2}\,\frac{1}{i\phi'}\,\frac{d}{dp} e^{i\phi}\,dp \\
&= -2\int_{-\infty}^{\infty} \frac{d}{dp}\frac{p}{(x+p^2)^3}\, e^{i\phi}\,dp = O(x^{-4}),
\end{aligned}$$

etc. Since we could go on like this forever, we see that $\mathrm{Ai}(x)$ *decreases faster than any (negative) power of* x as $x \to +\infty$.

This observation verifies that Ai is the solution of Airy's equation that decays rapidly at large x. On the other hand, we will show that *any* solution behaves in that limit as a linear combination of $x^{-1/4}e^{\pm 2x^{3/2}/3}$. So

$$\mathrm{Ai}(x) \sim Cx^{-1/4}e^{-2x^{3/2}/3}.$$

To determine C from the integral $(*)$ requires techniques of complex analysis that are beyond the scope of this course. (Answer: $C = \frac{1}{2\sqrt{\pi}}$.)

To establish the claim, carry out a calculation similar to that for the Bessel asymptotics, but without the i in the exponent: Write Airy's equation as $y'' = \omega^2 xy$. (Later we'll set $\omega = 1$, since we're more interested in large x than in large ω, but inserting the so-called "formal" parameter ω makes it easier to keep track of the relative sizes of the various terms.) Make the ansatz $y = Ae^{-S/\omega}$. Then

$$y'' = e^{-S/\omega}\left[\frac{1}{\omega^2}\,(S')^2 A - \frac{1}{\omega}(S''A + 2S'A') + A''\right],$$

and so the differential equation is equivalent to

$$0 = [(S')^2 - x]A - \omega(S''A + 2S'A') + \omega^2 A''.$$

This is satisfied to leading order if $S' = \pm\sqrt{x}$, or

$$S = \pm\frac{2}{3}x^{3/2}.$$

If we then let $A = A_0 + \omega^{-1}A_1 + \cdots$, we find to lowest order that $S''A_0 + 2S'A_0' = 0$, or

$$\frac{A_0'}{A_0} = -\frac{1}{2}\frac{S''}{S'} = -\frac{1}{4x}\,,$$

whence $\ln A_0 = -\frac{1}{4}\ln x$, or $A_0 = x^{-1/4}$. (As usual in such calculations, the constants of integration get absorbed into normalization constants multiplying the basis solutions we're constructing.) Finally, then, we arrive at

$$y(x) = x^{-1/4}e^{\pm 2x^{3/2}/3},$$

as predicted.

Turning points and WKB approximations in general

Consider a differential equation of the form

$$-\frac{d^2y}{dx^2} + \omega^2[V(x) - E]y = 0. \qquad (\dagger)$$

Many other linear second-order equations, including Bessel's, can be put into this form by change of variable. (In ($\dagger$) I'm using notation suggestive of quantum mechanics, except for using ω in the same role as elsewhere in these notes.) Suppose for definiteness that there is exactly one place, x_0, where $V(x_0) = E$, with

$$[V(x) - E]\ \begin{cases} < 0 & \text{for } x < x_0\,, \\ > 0 & \text{for } x > x_0\,. \end{cases}$$

For $x < x_0$ the solutions are oscillatory, and a phase-integral construction like the one for Bessel's equation shows that y is a linear combination of two solutions that are approximated for large ω by

$$[E - V(x)]^{-1/4}e^{\pm i\omega\int_0^x \sqrt{E - V(\tilde{x})}\,d\tilde{x}}. \qquad (\sharp)$$

For $x > x_0$ the solutions are (roughly) exponential in behavior, and a construction like that above for the Airy functions shows that y is a linear combination of two solutions that are approximated for large ω by

$$[V(x) - E]^{-1/4}e^{\pm\omega\int_0^x \sqrt{V(\tilde{x}) - E}\,d\tilde{x}}. \qquad (\flat)$$

One would like to know which linear combination of the solutions ($\sharp$) is the continuation of the exponentially decaying solution in ($\flat$).

For x near x_0 neither of the phase-integral solutions is accurate. (Indeed, $[V(x) - E]^{-1/4}$ goes to infinity at x_0 whereas the true solution must be well behaved.) This question has been answered by expanding $V(x)$ in a power series around x_0 and keeping just the first few terms. (We assume that V is differentiable at least a few times and that $V'(x_0) \neq 0$.) The solution ($\natural$) to this approximate equation is an Airy function (Ai with the variables suitably rescaled); it is a good approximation to y when $|x - x_0|$ is very small. From our previous discussion we know the asymptotic behavior of ($\natural$) in the two regions $x < x_0$ and $x > x_0$. These expressions can be "matched" to the phase-integral solutions to the true differential equation to find out how the solutions ($\flat$) "connect" to the solutions ($\sharp$). It isn't necessary to repeat this calculation every time; most quantum mechanics books will just tell you the answer: The decaying solution in ($\flat$) (with coefficient 1) evolves into the oscillatory solution

$$2[E - V(x)]^{-1/4} \cos\left(\omega \int_x^0 \sqrt{E - V(\tilde{x})}\, d\tilde{x} - \frac{\pi}{4}\right),$$

which matches our result for Airy's equation itself, where $\int_0^x \sqrt{E - V}\, d\tilde{x} = \frac{2}{3}(-x)^{3/2}$.