Math. 412 (Fulling) 19 October 2007

Test B – Solutions

Calculators may be used for simple arithmetic operations only!

When a question appears in two versions, answer the version appropriate to your status (honors or regular). Then work on the other version if you have time.

Famous integrals:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\, e^{-k^2 t}\, dk = \frac{1}{\sqrt{4\pi t}}\, e^{-x^2/4t}\,, \qquad \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\, e^{-|k|y}\, dk = \frac{1}{\pi}\,\frac{y}{x^2+y^2}\,.$$

1. *(35 pts.)* Solve by separation of variables or an equivalent transform technique:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (0 < x < \infty, \quad 0 < y < \infty),$$

$$\frac{\partial u}{\partial x}(0,y) = 0 \quad (0 < y < \infty),$$

$$u(x,0) = f(x) \quad (0 < x < \infty).$$

(Consider only bounded solutions.)

By separation of variables, or from general savvy about transforms, we see that the solution is a Fourier cosine transform in x:

$$u(x,y) = \int_0^\infty Y_k(y)\cos(ky)\, dk\,.$$

So, what is A?

Variable-separation thinking: The transverse part of the mode must satisfy $Y_k'' - k^2 Y_k = 0$, hence, to be bounded at $+\infty$, $Y_k(y) = A_k e^{-ky}$. A_k must be chosen to satisfy the initial condition, so it turns out to be

$$A_k = \frac{2}{\pi}\int_0^\infty \cos(kx) f(x)\, dx\,.$$

Transform thinking: Take the cosine transform of the PDE and the IC:

$$-k^2\,\hat{u} + \frac{\partial^2 \hat{u}}{\partial y^2} = 0\,, \qquad \hat{u}(k,0) = \hat{f}(k)\,.$$

The (bounded) solution is

$$\hat{u}(k,y) = \hat{f}(k) e^{-ky}\,,$$

where $\hat{f}$ is, up to your favorite normalization constant, the same as A_k earlier.

In either case, then,

$$u(x,y) = \int_0^\infty A_k e^{-ky}\cos(kx)\, dk\,,$$

with A_k as above.

2. *(30 pts.) (regular)* Find the Green function that gives the solution of Qu. 1 in the form

$$u(x,y) = \int_0^\infty G(x,z,y) f(z)\, dz\,.$$

(There are two methods. **Do** evaluate the integral if your method leads to one.)

Method 1: Use the results from Qu. 1; substitute

$$A_k = \frac{2}{\pi}\int_0^\infty \cos(kz) f(z)\, dz$$

into the u formula and strip off the z integral to get

$$G(x,z,y) = \frac{2}{\pi}\int_0^\infty e^{-ky}\cos(kx)\cos(kz)\, dk\,.$$

To evaluate the integral, extend it over the whole real line as

$$\frac{1}{\pi}\int_{-\infty}^\infty e^{-|k|y}\cos(kx)\cos(kz)\, dk$$

and use $\cos w = \frac{1}{2}(e^{iw} + e^{-iw})$ to get (after changing $k \to -k$ in half the terms)

$$\frac{1}{2\pi}\int_{-\infty}^\infty \left[e^{ik(x-z)} + e^{-ik(x+z)}\right] e^{-|k|y}\,.$$

Use the second of the famous integrals to write this as

$$G(x,z,y) = \frac{1}{\pi}\left[\frac{y}{(x-z)^2+y^2} + \frac{y}{(x+z)^2+y^2}\right].$$

Method 2: The Green function for Laplace's equation in the upper half plane is well known to be the second famous integral with x replaced by $x-z$. By the method of images for a Neumann boundary, the Green function for our problem is the sum of that with the same function evaluated at $x+z$ (see conclusion of the first method).

2. *(30 pts.) (honors)* Use a (well known) Green function to solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad (0 < x < \infty, \quad 0 < t < \infty),$$

$$\frac{\partial u}{\partial x}(0,t) = 0 \quad (0 < t < \infty),$$

$$u(x,0) = f(x) \quad (0 < x < \infty).$$

Method 1: The Green function for the heat equation on the whole real line is well known to be the first famous integral with x replaced by $x-z$. By the method of images for a Neumann boundary, the Green function for our problem is the sum of that with the same function evaluated at $x+z$:

$$G(x,z,t) = \frac{1}{\sqrt{4\pi t}}\left[e^{-(x-z)^2/4t} + e^{-(x+z)^2/4t}\right].$$

Method 2: Proceed as in Qu. 1 and Qu. 2-regular, Method 1: Solve the heat equation by separation of variables, plug the coefficient formula into the answer, and interchange the order of integrations to get

$$G(x,z,t) = \frac{2}{\pi}\int_0^\infty e^{-k^2 t}\cos(kx)\cos(kz)\, dk\,.$$

Do the trig and algebra to write this as the sum of two versions of the first famous integral.

3. *(35 pts.)* Construct the Green function that solves

$$y'' + 9y = f(x) \quad (0 < x < \pi),$$

$$y(0) = 0\,, \qquad y'(\pi) = 0\,.$$

Clearly state the formula for calculating y from G and f.

The Green function should satisfy

$$\frac{\partial^2 G}{\partial x^2} + 9G = \delta(x-z)\,, \tag{1}$$

$$G(0,z) = 0 = \frac{\partial G}{\partial x}(\pi, z)\,. \tag{2}$$

And (1) can be further explicated as

$$\frac{\partial^2 G}{\partial x^2} + 9G = 0 \quad (x \neq z), \tag{3}$$

$$G(z-\epsilon, z) = G(z+\epsilon, z)\,, \qquad \frac{\partial G}{\partial x}(z+\epsilon, z) - \frac{\partial G}{\partial x}(z-\epsilon, z) = 1\,. \tag{4}$$

A solution of (3) satisfying (2) must be of the form

$$G(x,z) = \begin{cases} A(z)\sin(3x) & (x < z), \\ B(z)\cos\big(3(\pi - x)\big) = -B\cos(3x) & (x > z). \end{cases}$$

Then from (4) we get

$$\sin(3z)A + \cos(3z)B = 0\,,$$
$$-3\cos(3z)A - 3\sin(3z)B = 1\,.$$

The solution of this system is

$$A = -\tfrac{1}{3}\cos(3z)\,, \qquad B = \tfrac{1}{3}\sin(3z)\,.$$

Thus

$$G(x,z) = \begin{cases} -\frac{1}{3}\sin(3x)\cos(3z) & (x < z), \\ -\frac{1}{3}\cos(3x)\sin(3z) & (x > z). \end{cases}$$

(The last line could also be written $+\frac{1}{3}\cos\big(3(\pi - x)\big)\sin(3z)$.)

The formula for the solution of the original problem is

$$y(x) = \int_0^\pi G(x,z) f(z)\, dz\,.$$

It could be written out as

$$y(x) = -\frac{1}{3}\int_0^x \cos(3x)\sin(3z) f(z)\, dz - \frac{1}{3}\int_x^\pi \sin(3x)\cos(3z) f(z)\, dz\,.$$