Math. 467 (Fulling) 11 October 2013

Midterm Test – Solutions

Name: ______________________________

1. *(Multiple choice – each 5 pts.) (Circle the correct capital letter.)*

   (a) Let P stand for the proposition "All right angles are congruent" and let Q stand for the SAS triangle congruence criterion. Which of the following is true?

   (A) Both P and Q were assumed as axioms by Euclid.
   (B) Both P and Q were proved as theorems by Euclid.
   (C) Q can easily be proved from P.
   (D) Modern geometry has Q as an axiom and P as a theorem, but in Euclid's writing it was the reverse.
   (E) Modern geometry has P as an axiom and Q as a theorem, but in Euclid's writing it was the reverse.

D

   (b) Which of these is **not** a theorem of Hilbert geometry (without a parallel postulate)?

   (A) If $\angle CAB < \angle DEF$ and $\angle DEF < \angle GHI$, then $\angle CAB < \angle GHI$.
   (B) If D is in the interior of $\angle CAB$, then so is every other point on $\overrightarrow{AD}$ except A.
   (C) Every point D in the interior of $\angle CAB$ lies on a segment joining a point E on $\overrightarrow{AB}$ to a point F on $\overrightarrow{AC}$.
   (D) If $\angle CAB < \angle DEF$ and $\angle DEF \cong \angle GHI$, then $\angle CAB < \angle GHI$.
   (E) If $\overrightarrow{AD}$ is between $\overrightarrow{AC}$ and $\overrightarrow{AB}$, then $\overrightarrow{AD}$ intersects segment BC.

C (This is the "warning" on p. 115.)

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2. *(16 pts.)* Prove **ONE** of the triangle congruence theorems, either ASA or SSS. *(At most 8 points extra credit for doing both.)*

For ASA, see solution to Exercise 3.26, pp. 151–152. SSS was extensively discussed in class.

3. *(20 pts.)* The *tetrahedron* is an incidence geometry consisting of 4 points and 6 lines. (The lines are the two-point subsets.)

   (a) State the 3 incidence axioms that this system (like every incidence geometry) satisfies.

See Chapter 2.

   (b) State the parallelism property that this particular system satisfies.

Euclidean (see pp. 74–75).

4. *(11 pts.)* Simplify $\neg\forall x\,\exists y\,\big[[(x<y)\wedge\exists z\,(x+z>y)]\ \Rightarrow\ (x+y\le 3)\big]$.
(Push the "¬" in as far as you can! Truth or falsity of the statement is irrelevant. In Greenberg's notation, ¬ is ∼, and ∧ is &.)

Work in steps:

$$\exists x\,\forall y\,\neg\big[[(x<y)\wedge\exists z\,(x+z>y)]\ \Rightarrow\ (x+y\le 3)\big]\,;$$

$$\exists x\,\forall y\,\big[[(x<y)\wedge\exists z\,(x+z>y)]\wedge\neg(x+y\le 3)\big]\,;$$

$$\exists x\,\forall y\,\big[(x<y)\wedge\exists z\,(x+z>y)\wedge(x+y>3)\big]\,.$$

5. *(18 pts.)* *Pasch's theorem* says, loosely, that any line that goes into a triangle comes back out.

   (a) State the theorem precisely.

   (b) Prove it.

See p. 114.

6. *(Essay – 25 pts.)* Let $\mathcal{S}$ be a set. Recall that an *equivalence relation* on $\mathcal{S}$ is a binary relation $\sim$ with the properties (for all A, B, C in $\mathcal{S}$)
   (1) reflexivity: $A\sim A$,
   (2) symmetry: $A\sim B\ \Rightarrow\ B\sim A$,
   (3) transitivity: $A\sim B\wedge B\sim C\ \Rightarrow\ A\sim C$.
   Recall also that a *partition* of $\mathcal{S}$ is a collection $\{S_i\}$ of subsets of S with the properties
   (4) disjointness: $S_i\cap S_j=\emptyset$ if $i\ne j$,
   (5) exhaustiveness: $S=\bigcup_i S_i$.

   Show how every equivalence relation on $\mathcal{S}$ determines a partition of $\mathcal{S}$, and how every partition of $\mathcal{S}$ determines an equivalence relation on $\mathcal{S}$. (In each of the two cases, explain how the new entity is defined from the old one and prove that it satisfies all the clauses in the definition.)