Math. 467 (Fulling) 11 March 2021

Midterm Test – Solutions

Name: ______________________________

1. *(Multiple choice – each 5 pts.) (Circle the correct capital letter.)*

   (a) Euclid's geometry is based on five fundamental, undefined terms, but the importance of one of them was not fully realized until the work of Pasch and Hilbert over 2000 years later. Which one?

   (A) point
   (B) line
   (C) lie on
   (D) between
   (E) congruent

D

   (b) Which of these statements is **false**? A right angle

   (A) is congruent to its supplement.
   (B) is defined as the space between two perpendicular rays.
   (C) has angle measure 90°.
   (D) is congruent to every other right angle.
   (E) can be constructed by straightedge and compass at any given point on a given line.

B (An angle is defined as a set of two rays, not the space between them.)

   (c) The simplest models of incidence geometry have finitely many points, and the lines are all the 2-point subsets. Among these the model with 5 points is significant because

   (A) It exhibits the Euclidean parallelism property.
   (B) It is the largest model that exhibits the elliptic parallelism property.
   (C) It is the smallest model that exhibits the hyperbolic parallelism property.
   (D) It has none of the three classic parallelism properties.

C

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2. *(12 pts.)* Simplify $\neg\forall x\,\exists m\,\big[x<0\vee S(m,x)\;\Rightarrow\;\exists n\,\big(T(m,n)\wedge n>x\big)\big]$.
   (Push the "$\neg$" in as far as you can!) (In Greenberg's notation, $\neg$ is $\sim$, and $\wedge$ is &.)

$$\exists x\,\forall m\neg\big[x<0\vee S(m,x)\;\Rightarrow\;\exists n\,\big(T(m,n)\wedge n>x\big)\big]\,.$$

$$\exists x\,\forall m\,\Big[\big(x<0\vee S(m,x)\big)\wedge\neg\big(\exists n\,\big(T(m,n)\wedge n>x\big)\big)\Big]$$

$$\exists x\,\forall m\,\Big[\big(x<0\vee S(m,x)\big)\wedge\forall n\,\neg\big(T(m,n)\wedge n>x\big)\Big]$$

$$\exists x\,\forall m\,\Big[\big(x<0\vee S(m,x)\big)\wedge\forall n\,\big(\neg T(m,n)\vee n\le x\big)\Big]$$

3. *(18 pts.)* State the three *congruence* axioms that **don't** involve angles. (These are the first three.)

[See pp. 119–120.]

4. *(15 pts.)* State the *crossbar theorem* and draw a sketch to illustrate it.

[See p. 116.]

5. *(20 pts.)* Do **ONE** of these [(A) or (B)]. *(Extra credit for doing both is limited to 10 points.)*

   (A) Use Proposition 2.3 (restated in Question 6, below) to give a quick proof of the "Useful Lemma" ("For every point, P, there are two points, A and B, that are not collinear with P.").

Given P, let L be any line through P. (If you want to be really picky, appeal to I-3 to prove that there are points distinct from P and then appeal to I-1 to construct a line through P.) Let A be any point that is not on L (guaranteed by 2.3). Let B be any point (besides P) that is on L (guaranteed by I-2). Because L is the unique line through P and B (by I-1), A is not collinear with P and B.

   (B) Recall that Axiom B-4 states:

   For any line l and any three points A, B, C not lying on l,

   (i) If A and B are on the same side of l and if B and C are on the same side of l, then A and C are on the same side of l.

   (ii) If A and B are on opposite sides of l and if B and C are on opposite sides of l, then A and C are on the same side of l.

   Prove the Corollary:

   (iii) If A and B are on opposite sides of l and if B and C are on the same side of l, then A and C are on opposite sides of l.

[See notes, pp. 23–24 of the printer version. Easier than remembering the details of the propositional calculus given there is the following reductio ad absurdum:]

Assume B and C are on the same side and A and C are on the same side. Then (i) with the letters changed implies that A and B are on the same side. This contradicts the other half of the hypothesis of (iii), so A and C must be on opposite sides. (Here we have tacitly used the facts that "same side" is a symmetric relation and that "opposite side" is exactly the negation of "same side".)

6. *(Essay – 20 pts.) IMPROVEMENTS IN THE REWRITE GET ONLY HALF CREDIT. E.G., IF YOUR INITIAL SCORE IS 16, YOUR FINAL SCORE WON'T EXCEED 18.*

   Proposition 2.3 is, "For every line, there is at least one point not lying on it." One of your teammates has proposed the following proof:

   > According to Axiom I-3, there are three points (call them A, B, and C) such that no line is incident with all of them. Let l be the line $\overleftrightarrow{AB}$. Then C does not lie on l, QED.

   Explain what is wrong with this proof. Then give a correct proof. (Don't use the "Useful Lemma" in your proof.)