

Minakshisundaram and the Birth of Geometric Spectral Asymptotics

Those who do not know what “geometric spectral asymptotics” is should think in terms of this elementary example: A vibrating string of length L has normal modes (eigenfunctions) $\phi_n(x) = \sin(n\pi x/L)$ with frequencies $\omega_n \equiv \sqrt{\lambda_n} = n\pi/L$ (in units where the speed of sound is 1), where n runs through all the positive integers. Note that

$$Z(s) \equiv \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s}$$

is essentially the famous Riemann zeta function. The eigenfunctions satisfy the equation and boundary condition

$$\phi_n'' = -\lambda_n \phi_n, \quad \phi_n(0) = \phi_n(L) = 0.$$

If we changed the boundary condition to $\phi_n'(0) = \phi_n'(L) = 0$ (which would be appropriate for sound waves in a pipe with open ends), the normal modes would become $\phi_n(x) = \cos(n\pi x/L)$ and the case $n = 0$ would need to be included. Although we started with a wave problem, exactly the same eigenfunctions solve the problem of heat conduction in a bar of length L . The Green function (integral kernel) that gives the temperature distribution $u(x)$ at time t in terms of the initial temperature $f(y)$ at time 0 is

$$K(t, x, y) = \sum_n \frac{1}{\|\phi_n\|^2} \phi_n(x) \phi_n(y) e^{-\lambda_n t}. \quad (1)$$

On the other hand, by the method of images (and the well known solution of the heat problem on the whole real line) one can see that

$$K(t, x, y) = \frac{1}{\sqrt{4\pi t}} \left[e^{-(x-y)^2/4t} \pm e^{-(x+y)^2/4t} \pm e^{-(x-2L+y)^2/4t} + \dots \right], \quad (2)$$

where the $+$ sign applies to the Neumann (organ pipe) boundary condition and the $-$ to the Dirichlet (string) case. Now examine the integral $\int_0^L K(t, x, x) dx$ at small t ; it can be shown that only the first term in (2) makes a significant contribution except when x is close to an endpoint, whereupon the second or third term must be taken into account; and one gets

$$\int_0^L K(t, x, x) dx \sim \frac{L}{\sqrt{4\pi t}} \pm \frac{1}{2}. \quad (3)$$

Therefore, from a knowledge of the small-time asymptotics (3) of K we can read off the size of the interval, L , and also (from the second term) which boundary condition is in force. Working in another direction, comparing (3) with (1) one could get some information about the eigenvalues if one did not know them already; for example, the fact that expressions (3) for the Neumann and Dirichlet cases differ by exactly 1 reflects the presence of the extra eigenvalue, $\lambda_0 = 0$, in the Neumann case.

Geometric spectral asymptotics is the extension of these considerations to more complicated, multidimensional differential operators, such as

$$H = -\frac{\partial}{\partial x} a(x, y) \frac{\partial}{\partial x} - \frac{\partial}{\partial y} b(x, y) \frac{\partial}{\partial y} + c(x, y).$$

On one side is the “geometrical” information contained in the operator itself, including the region in (x, y) space where it acts, and the boundary conditions if any, as well as the coefficient functions (here a, b, c). On the other side is the “spectral” information, such as eigenvalues and eigenfunctions satisfying $H\phi_n = \lambda_n\phi_n$. (Henceforth we also assume without loss of generality that $\|\phi_n\| = 1$ and $\phi_n(x)$ is real.) In the middle sit the various integral kernels that solve various differential equations involving H , or, more generally, represent functions of H ; the heat kernel K , related to the eigenfunctions by (1), is one of these. The study of these kernels, usually in various asymptotic limits, is the principal tool for relating the geometry to the spectrum.

Around 1950, S. Minakshisundaram made three major contributions to the creation of this subject.

In his paper [1] he generalized the Riemann zeta function by studying

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^s} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(y)}{\lambda_n^s},$$

the second of these being the kernel of the operator H^{-s} . ([1] does this for the Laplacian acting in a region in \mathbf{R}^n , but the idea applies to more general problems.) As functions of the complex variable s , these zeta functions have nice analytic properties, and the residues at their poles give information about the averaged density of the eigenvalues in the high-frequency limit, and also about the eigenfunctions. Minakshisundaram credits T. Carleman (in 1934) with the germ of the idea, but it seems that [1] and [2] are the papers that made generalized zeta functions into standard tools.

Minakshisundaram’s most famous paper [2], written in collaboration with Å. Pleijel, deals with the generalization of the Laplacian operator to Riemannian manifolds. In the “flat” cases in [1] he had used the heat kernel (1) as an intermediate step in calculating the zeta functions, or at least proving their analytic properties. On a curved manifold one can’t expect to find exact formulas like (2), but in [2] the authors showed that a local approximation to K can be constructed by a recursive method first applied by J. Hadamard for solving the wave equation. (Nowadays the calculations are done in greater generality and efficiency by pseudodifferential operators — i.e., introducing a local Fourier transform.) [2] took a detour through yet another Green function, but in a less well known paper [3] Minakshisundaram showed how to go directly from K to zeta. For a variety of reasons the heat kernel is especially well adapted to deriving the main facts of spectral asymptotics — the relation between the coarse features of the geometry and those of the spectrum. Therefore, [2] founded an industry. Important extensions and applications were made by the mathematicians Kac, Singer, Seeley, Gilkey, Atiyah, and Patodi, just to mention a few. Related calculational methods were introduced into quantum field theory by the physicists Schwinger and DeWitt.

The analysis of the heat kernel yields a power series in \sqrt{t} for K or its integral, which had only two terms in our example (3) but is infinite in general. The coefficients in the series can be expressed fairly simply in terms of the geometry of the problem, and they are closely related to the poles of the corresponding zeta function. Another related series — the inverse Laplace transform of the heat-kernel expansion — is a formal expansion (in negative powers of λ) for the density of eigenvalues in the high-frequency limit. Unfortunately, in general only the leading term of this series provides a literally valid asymptotic formula for the spectral density. Since the eigenvalues form a discrete sequence, such a series could not be true to all orders; in fact, it usually goes bad even sooner, because of oscillations in the density (which make no contribution to the asymptotics of K , but swamp the power-law terms in the putative asymptotics of the spectrum). So, are the higher-order terms in the series just nonsense? Here is where Minakshisundaram’s third contribution comes in: his book [4] written with K. Chandrasekharan, which is one step in the development of the theory of Riesz means — a generalization of the Cesàro summation (averaging of partial sums) used to make sense of a divergent Fourier series. In the late 1960s L. Hörmander showed that Riesz means provide a convenient, systematic way of interpreting all the terms in the inverse Laplace transform of the heat expansion as correct and precise descriptions of the “averaged” behavior of the spectral density.

Professor Minakshisundaram did not live to see the explosive developments in spectral asymptotics in the early 1970s. First, there was a huge upsurge of applications to differential geometry, analysis, general relativity, gauge theory, quantum mechanics, and statistical mechanics. Second, a new chapter was opened by the work of Gutzwiller, Balian, Colin de Verdière, and many other mathematicians and physicists. As previously remarked, the “classic” spectral asymptotics based on the local construction of the heat kernel yields only the averaged behavior of the spectrum, not its oscillatory fine structure. The “new” theory, still under development, relates that fine structure to a corresponding fine structure of the geometry, namely, the periodic orbits of the underlying classical-mechanical system having H as quantum Hamiltonian. This connection is relevant to the shell structure of nuclei, the electromagnetic vacuum energy discovered by Casimir, and the distinction between chaotic and integrable systems. The link between geometry and spectrum is still usually sought through integral kernels, but one must study kernels that are harder to calculate than K , precisely because they do not separate the coarse features from the fine structure.

References

1. A generalization of Epstein zeta functions, *Canad. J. Math.* **1** (1949), 320–327.
2. Some properties of the eigenfunctions of the Laplace-operator on Riemannian manifolds, *Canad. J. Math.* **1** (1949), 242–256.
3. Eigenfunctions on Riemannian manifolds, *J. Indian Math. Soc.* **17** (1953), 159–165.
4. *Typical Means*, Oxford U. Press, 1952.

S. A. Fulling
 Depts. of Mathematics and Physics
 Texas A&M University
 College Station, Texas, USA