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ENERGY DENSITY AND PRESSURE IN POWER-WALL MODELS

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A finite ultraviolet cutoff near a reflecting boundary yields a stress tensor that violates the basic energy-pressure relation. Therefore, a "soft" wall described by a power-law potential, which needs no ad hoc cutoff, is being investigated by the collaboration centered at Texas A&M University and the University of Oklahoma. Progress is reported here.

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1. Hard Walls: The Pressure Paradox

1.1. *Basic formalism and empty space*

We model the electromagnetic vacuum-energy problem by a massless scalar field in three space dimensions, subjected to the Dirichlet boundary condition on the "conducting" boundaries. The vacuum energy can be calculated from a Green function, such as the "cylinder kernel,"[1–3]

$$\overline{T}(t,\mathbf{r},\mathbf{r}') = -\sum_{n=1}^{\infty} \frac{1}{\omega_n}\, \phi_n(\mathbf{r})\phi_n(\mathbf{r}')^* e^{-t\omega_n} \,, \tag{1}$$

where the ϕ_n are the normalized eigenfunctions with frequencies ω_n. The classical formulas for the components of the vacuum expectation value of the stress-energy tensor are

$$\mathcal{E} = T_{00} = -\frac{1}{2}\frac{\partial^2 \overline{T}}{\partial t^2} + \beta \nabla_{\mathbf{r}} \cdot \left[\nabla_{\mathbf{r}'} \overline{T}\right]_{\mathbf{r}'=\mathbf{r}}, \tag{2}$$

$$\begin{aligned} p_j = T_{jj} = {} & \frac{1}{8}\left(\frac{\partial^2 \overline{T}}{\partial x_j^2} + \frac{\partial^2 \overline{T}}{\partial {x'_j}^2} - 2\frac{\partial^2 \overline{T}}{\partial x_j\,\partial x'_j}\right)_{\mathbf{r}'=\mathbf{r}} \\ & + \beta\left(\frac{\partial^2 \overline{T}}{\partial t^2} + \frac{1}{2}\frac{\partial^2 \overline{T}}{\partial x_j^2} + \frac{1}{2}\frac{\partial^2 \overline{T}}{\partial {x'_j}^2} - \sum_{k\neq j}\frac{\partial^2 \overline{T}}{\partial x_k\,\partial x'_k}\right)_{\mathbf{r}'=\mathbf{r}}. \end{aligned} \tag{3}$$

Here $\beta = \xi - \frac{1}{4}$ is the curvature-coupling parameter ($\beta = -\frac{1}{4}$ for minimal coupling, $\beta = -\frac{1}{12}$ for conformal coupling). In this exposition we normally take $\beta = 0$ ($\xi = \frac{1}{4}$) and describe the minor changes needed for other values of ξ as afterthoughts.

In infinite empty space the cylinder kernel is

$$\overline{T}_0(t-t', \mathbf{r}, \mathbf{r}') = -\frac{1}{2\pi^2}\frac{1}{(t-t')^2 + \|\mathbf{r}-\mathbf{r}'\|^2}. \tag{4}$$

Note that t is playing a dual role: It may be thought of as the parameter of an ultraviolet cutoff as in (1), or as a Wick rotation of the time coordinate: $t - t' = -i(x^0 - x'^0)$. Indeed, the ultraviolet cutoff can be generalized to "point-splitting" regularization, where $\overline{T}$ (or, historically more often, a Green function for the wave equation) and its derivatives in (2) and (3) are expanded in powers of the displacement $tu^\nu \equiv x^\nu - x'^\nu$ (historically, its generalization to curved space–time as a tangent vector). The result in curved space–time (with signature $g_{00} < 0$) is[4]

$$T_{\mu\nu} = \frac{1}{2\pi^2 t^4}\left(g_{\mu\nu} - 4\frac{u_\mu u_\nu}{u_\rho u^\rho}\right) \tag{5}$$

for the leading term, which is the entirety when $\overline{T} = \overline{T}_0$. Equation (5) reduces to

$$T_{\mu\nu} = \frac{1}{2\pi^2 t^4}\,\mathrm{diag}(3,1,1,1) \tag{6}$$

if one returns to the original, purely temporal cutoff. On the other hand, if the direction-dependent term in (5) is simply ignored, the result is

$$T_{\mu\nu} = \frac{1}{2\pi^2 t^4}\,\mathrm{diag}(-1,1,1,1). \tag{7}$$

In studies of vacuum energy it is customary to discard all of (6) on the grounds that it is ubiquitous and, therefore, unobservable. In quantum theory in curved space–time, however, one can argue that it is more in keeping with standard renormalization theory to retain (7) with the divergent prefactor replaced by an arbitrary but finite renormalized constant; this is the result that would be produced by Pauli–Villars regularization.[5] This term amounts to a renormalization of the cosmological constant in Einstein's equation.

1.2. *Flat hard walls*

When a plane boundary is present, the kernel can be constructed by the method of images. Since $\overline{T}_0$ contributes only the zero-point energy of infinite space, which is either unobservable or interpretable as a cosmological constant, the term relevant to the calculation is the image term, and we henceforth understand $\overline{T}$ to mean this "renormalized" kernel. If the perfectly reflecting boundary is at $z = 0$, it is

$$\overline{T} = \frac{1}{2\pi^2} \frac{1}{t^2 + (\mathbf{r}_\perp - \mathbf{r}'_\perp)^2 + (z+z')^2} \qquad (\mathbf{r}_\perp = (x,y))\,. \tag{8}$$

There is no loss of generality in assuming $\mathbf{r}'_\perp = 0$ (and $t' = 0$). At present t, $\mathbf{r}_\perp$, and $z - z'$ are all still available as cutoff parameters. Let

$$M = t^2 + x^2 + y^2 + (z+z')^2\,. \tag{9}$$

From (2) and (3) one finds the energy density and pressure for $\xi = \frac{1}{4}$,

$$2\pi^2 \mathcal{E} = M^{-3}[-3t^2 + x^2 + y^2 + (z+z')^2]\,, \tag{10}$$

$$2\pi^2 p_1 = M^{-3}[-t^2 + 3x^2 - y^2 - (z+z')^2]\,, \tag{11}$$

and the correction terms for $\xi \neq \frac{1}{4}$,

$$2\pi^2 \Delta\mathcal{E} = \beta M^{-3}[-4t^2 - 4x^2 - 4y^2 + 12(z+z')^2]\,, \tag{12}$$

$$2\pi^2 \Delta p_1 = \beta M^{-3}[4t^2 + 4x^2 + 4y^2 - 12(z+z')^2]\,. \tag{13}$$

Of course, the formulas for p_2 and Δp_2 are precisely analogous. The formulas for p_3 and Δp_3 are identically 0, as they should be because a rigid displacement of the wall perpendicularly to itself would not change the total energy (even when only one side of the wall is considered).

The interesting question arises when one imagines another planar boundary perpendicular to the first, say at $x = 0$, and calculates the pressure on that boundary from (11), considering one side of the boundary only (Fig. 1). For definiteness consider the case that the energy density is positive. If we think of the entire system as enclosed in a large stationary box, then a displacement of the test boundary

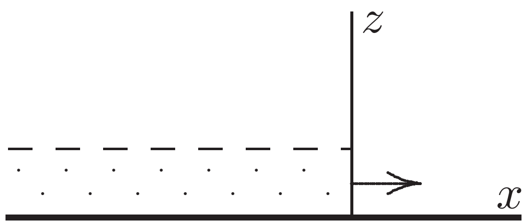

Fig. 1. The total energy in the dotted region created by the horizontal wall depends linearly on the position of the movable vertical wall. (The surface energy of the vertical wall, and the edge energy of the corner junction, are irrelevant to the calculation. The energies on the right side of the vertical wall are not considered here, but see Subsec. 1.4.)

to $x > 0$ would enlarge the volume of space occupied by the energy density (10), and hence the total energy would increase. This energy change must come from the force on the boundary associated with the pressure p_1. In accordance with the principle of energy balance, or virtual work, one therefore expects

$$F = \int_0^\infty T^{11}\,dz = -E = -\int_0^\infty T^{00}\,dz\,. \tag{14}$$

(E is an energy per unit area in the (x, y) plane; F is a force per unit distance in the y direction. The minus sign appears because F is the force exerted *by* the quadrant of space considered, not *on* it. In other words, the region wants to shrink, so the pressure is negative.)

We observe that $\Delta p = -\Delta\mathcal{E}$ already at the integrand level. (Furthermore, their integrals are individually zero, as naively expected since that part of T^{00} is a total divergence.) In the main terms, (10) and (11), if one removes the cutoffs by setting $t = x = y = 0$, $z' = z$, then the integrands are again equal,

$$\mathcal{E} = \frac{1}{32\pi^2 z^4} = -p_1\,, \tag{15}$$

so (14) is formally satisfied, but the integrals are divergent.

The traditional ultraviolet cutoff corresponds to $x = y = 0$, $z' = z$, but $t > 0$. Thus $M = t^2 + 4z^2$ and

$$2\pi^2\mathcal{E} = M^{-3}(-3t^2 + 4z^2), \quad 2\pi^2 p_1 = M^{-3}(-t^2 - 4z^2) = -(t^2 + 4z^2)^{-2}.$$

If one integrates over z from 0 to ∞ before removing the cutoff, one finds

$$F = +\tfrac{1}{2}E$$

instead of (14). This is the "pressure paradox."[6]

Remark. (i) The numerical value of E in this calculation is negative and is the same as one would get from the Laurent series of the total energy sum with ultraviolet cutoff,

$$E = \frac{1}{2}\sum_n \omega_n e^{-t\omega_n},$$

normalized to unit area in the infinite-volume limit. (ii) If the wall at $x = 0$ is itself a conductor, the full Green function of the problem contains two additional image terms, but they do not affect the conclusions. (iii) If one attempted this calculation with the wave kernel, it would fail at an even earlier stage, since the supposedly regularized energy density would have a singularity in the physical region (at $z = \frac{t}{2}$). (iv) Equation (14) is not an automatic consequence of the local energy-momentum conservation law, $\partial_\mu T^{\mu\nu} = 0$, inside the cavity. The latter is satisfied, even in the presence of a finite cutoff.[7]

Whatever the case may have been for the original spectral sum (1), in the explicit Green function (8) point separation in any direction is an equally effective regularization. We therefore examine the effects of varying the direction. If $t = 0 = y$ and $z' = z$, but $x \neq 0$, then t and x exchange roles in the foregoing calculation, and

$$F = 2E\,.$$

If the points are separated in the neutral (y) direction, $t = 0 = x$, $z' = z$, we get $M = y^2 + 4z^2$,

$$2\pi^2\mathcal{E} = (y^2 + 4z^2)^{-2}\,, \quad 2\pi^2 p_1 = -(y^2 + 4z^2)^{-2}\,. \tag{16}$$

After integrating, therefore, we have $F = -E$ (with a positive value for E), which is what *should* happen (14). (Separation in the z direction leads to some pathologies and ambiguities, so we do not consider it.)

1.3. *Possible responses to the pressure paradox*

(i) It can be argued that divergent terms are so cutoff-dependent that they have no physical meaning whatsoever, and the only meaningful calculations are those in which such terms can be canceled out (e.g. forces between rigid bodies).

(ii) In Ref. 7 we argued that expressions with finite cutoff, such as $2\pi^2\mathcal{E} = (y^2 + 4z^2)^{-2}$ (where y is now a cutoff parameter, not a coordinate) can be regarded as ad hoc models of real materials, more physical and instructive than their limiting values, such as $\mathcal{E} = 1/32\pi^2 z^4$. But the paradox casts some doubt on the viability of this point of view. It now appears that physically plausible results can be obtained only by using different cutoffs for different parts of the stress tensor. For the leading divergence (and higher-order divergences in the bulk that occur in curved space–time or external potentials) the preferred ansatz is "covariant point-splitting"[4] based on the wave kernel, treating all directions in space–time equivalently, and removing the cutoff-dependent terms in such a way that the only ambiguity remaining can be regarded as a renormalization of the cosmological constant. For the divergences at boundaries, it appears that the points must be separated parallel to the boundary, but in a direction orthogonal to the component of the stress tensor being calculated. Moreover, if the separation has a time component, a Wick rotation seems mandatory. This situation cannot be regarded as a logically sound, long-term solution; its sole justification is that, unlike less contrived alternatives, it does not immediately produce results that are obviously wrong.

(iii) One should find a better model! That is the approach followed in the rest of this presentation. We seek to modify the simple linear Dirichlet theory in a minimal way that will replace the divergent energies and pressures with finite ones, without introducing any ad hoc cutoffs that can disrupt the proper relationship between energy and pressure. The original theory should be recovered in some limit, and we hope that improved physical understanding of the divergences will result.

1.4. *Curved hard walls*

It may be objected that the discussion above is entirely "scholastic", because there is an equal and opposite force from the other side of the movable wall that renders the paradoxical force unobservable even in principle. However, recall that the paradox was discovered[6,8] in calculations for a spherical boundary. There also the ultraviolet cutoff gave $F = +\frac{1}{2}E$. But in that case the inside and outside energy layers have the same sign; there is a total energy proportional to surface area, and no cancellation. Therefore, the pressure paradox cannot be dismissed. Presumably the same thing happens for a cylindrical boundary; that case is under investigation.[9]

2. Flat Soft Walls

2.1. *Precursors*

An ambitious approach to a realistic theory of a conductor is the plasma model of Barton.[10–13] There the charges in the conducting medium are quantized as a separate physical system interacting nonlinearly with the quantized field. Here we prefer to remain within the more elementary framework of linear quantum field theory, at the cost of lesser empirical relevance, in hopes of casting light on the original hard-boundary theory — perhaps salvaging it in some way.

The closest work that we know of in previous literature is that of Actor and Bender,[14] in which the perfectly reflecting wall is replaced by a harmonic-oscillator potential. That paper was written before the modern critiques of formal renormalization[15,16] and the modern emphasis on local quantities (such as energy density). It deals with total energies calculated by zeta-function regularization.

Other similar work includes a paper of Bordag[17] and the large number of papers by Graham, Jaffe, Olum, and coworkers (such as Refs. 16 and 18–20) culminating in the book of Graham, Quandt, and Weigel.[21] The latter program differs from ours superficially by dealing with a high, narrow potential hill (instead of a one-sided wall), and more fundamentally in the choice of calculational methods (techniques of scattering theory instead of local, differential-equation analyses), which results in a rather different point of view.

2.2. *The power wall*

We consider a scalar wave equation

$$\Box\,\varphi = v\varphi \tag{17}$$

on all of three-dimensional space, with a potential (or spatially varying Klein–Gordon mass-squared) of the form

$$v(\mathbf{r}) = \begin{cases} 0, & z < 0\,, \\ z^\alpha, & z > 0\,. \end{cases} \tag{18}$$

Note that $v(1) = 1$ for all α. The potential represents an increasingly steep wall near $z = 1$ as α increases to ∞.

The problem can be solved by separation of variables. The eigenfunctions are

$$\phi_{(\mathbf{k}_\perp, p)} = (2\pi)^{-1} e^{i\mathbf{k}_\perp\cdot\mathbf{r}_\perp}\phi_p(z)\,, \tag{19}$$

where

$$\left(-\frac{\partial^2}{\partial z^2} + v(z) - p^2\right)\phi_p(z) = 0\,. \tag{20}$$

Thus, with $\omega^2 = \mathbf{k}_\perp{}^2 + p^2$ and ϕ_p properly normalized, we have

$$\overline{T}(t, \mathbf{r}_\perp, z, z') = \frac{-1}{(2\pi)^2}\int_{\mathbf{R}^2} d\mathbf{k}_\perp \int_0^\infty dp\, \frac{e^{-\omega t}}{\omega}\, e^{i\mathbf{k}_\perp\cdot\mathbf{r}_\perp}\phi_p(z)\phi_p(z') \tag{21}$$

$$= \frac{-2}{(2\pi)^3}\int_{\mathbf{R}^3} d\omega\, d\mathbf{k}_\perp \int_0^\infty dp\, \frac{e^{i(\omega t + \mathbf{k}_\perp\cdot\mathbf{r}_\perp)}\phi_p(z)\phi_p(z')}{\omega^2 + \mathbf{k}_\perp{}^2 + p^2}\,. \tag{22}$$

Here $\mathbf{r}_\perp$ may be replaced by $\mathbf{r}_\perp - \mathbf{r}'_\perp$ whenever $\mathbf{r}'_\perp$ derivatives are needed.

When $z < 0$, one has

$$\phi_p(z) = \sqrt{\frac{2}{\pi}}\,\sin[pz - \delta(p)] \tag{23}$$

for some real phase shift, $\delta(p)$. When $z > 0$, the solution can be expressed in terms of standard special functions in at least two cases:

$$\phi_p(z) \propto \begin{cases} \mathrm{Ai}(z - p^2), & \alpha = 1\,, \\ D_{\frac{1}{2}(p^2-1)}(\sqrt{2}\,z), & \alpha = 2\,. \end{cases} \tag{24}$$

The solutions on the two sides are related by

$$\tan\big(\delta(p)\big) = -p\,\frac{\phi_p(0)}{\phi'_p(0)} \tag{25}$$

and a complicated formula for the normalization constant in Eq. (24) that need not concern us now.

2.3. *The Texas approach*

The construction of a Green function in a separable problem depends on the order in which the separated variables are tackled. Details of our first approach have been given in Ref. 22, so we shall be brief here.

The asymptotic behaviors of $\delta(p)$ are easily determined. For example, for $\alpha = 1$ (the Airy function).

$$\delta(p) \sim \begin{cases} p\,3^{2/3}\Gamma(\frac{4}{3})/\Gamma(\frac{2}{3}), & p \to 0\,, \\ \frac{2p^3}{3} + \frac{\pi}{4}\,, & p \to +\infty\,. \end{cases} \tag{26}$$

In general, $\delta(p) \propto p^{1+2/\alpha}$ as $p \to +\infty$ and $\delta(p) \sim c_1 p + c_2 p^3$ as $p \to 0$.

Remark. The power wall potential is of no particular experimental interest; it was chosen for analytical convenience. Any potential with similar qualitative properties would do as well for our purposes. Since the induced stress outside the wall depends only on δ, the possibility arises of parametrizing wall models by $\delta(p)$ instead of $v(z)$. Unfortunately, we do not yet know what conditions to impose on δ to ensure that it arises from a steeply rising potential (or any potential at all).

After integration over the transverse Fourier dimensions in (21), one arrives at

$$\overline{T}_{\rm ren} = \frac{1}{2\pi^2}\int_0^\infty dp\,\frac{e^{-sp}}{s}\,\cos\bigl(p(z+z') - 2\delta(p)\bigr) \tag{27}$$

in the potential-free region ($z < 0$). Here $s \equiv \sqrt{t^2 + |\mathbf{r}_\perp|^2}$. The components of $T^{\mu\nu}$ are second derivatives of $\overline{T}_{\rm ren}$.

The convergence of integral (27) is extremely delicate when $s \to 0$, which is precisely where we need it. (In fact, the convergence is only in a distributional sense.) A slight improvement is attained by going to polar coordinates in Fourier space:

$$\overline{T}_{\rm ren} = \frac{1}{\pi^3}\int_0^\infty d\rho \int_0^1 du\, s^{-1}\sin(s\rho\sqrt{1-u^2}\,)\cos\bigl((z+z')\rho u - 2\delta(\rho u)\bigr)\,. \tag{28}$$

The differentiated integrals for $T^{\mu\nu}$ are, of course, even worse. Such integrals are being investigated with Riesz–Cesàro summation and modern methods for oscillatory quadrature (see, e.g. Ref. 23), and preliminary results (for $\alpha = 1$) look plausible.[24]

2.4. *The Oklahoma approach*

The Texas approach in effect did a generalized Fourier analysis in z to get to the problem of a *reduced Green function* in the $(t, \mathbf{r}_\perp)$ coordinates. (Note that

$$\frac{e^{-sp}}{s}\,, \quad s = \sqrt{t^2 + |\mathbf{r}_\perp|^2}\,, \tag{29}$$

is a Yukawa potential.) The oscillations in the eigenfunctions $\phi_p(z)$ are the source of the bad integral behavior.

Instead, let us do a Fourier analysis in the transverse dimensions (including t) to define a reduced Green function in the z direction.[25] It will vanish at infinity, not oscillate. In other words, in place of (21) we arrive at

$$\overline{T}(t, \mathbf{r}_\perp, z, z') = \frac{-2}{(2\pi)^3}\int_{\mathbf{R}^3} d\omega\, d\mathbf{k}_\perp\, e^{i(\omega t + \mathbf{k}_\perp\cdot\mathbf{r}_\perp)} g_{\omega,\mathbf{k}_\perp}(z, z')\,. \tag{30}$$

(The notation in Ref. 25 is slightly different.)

For $\alpha = 1$, in the region $z, z' < 0$, and with

$$\kappa \equiv \sqrt{\mathbf{k}_\perp{}^2 - \omega^2}\,, \tag{31}$$

one finds

$$g_{\omega,\mathbf{k}_\perp}(z, z') = \frac{1}{2\kappa}\,e^{-\kappa|z-z'|} + \frac{1}{2\kappa}\,e^{\kappa(z+z')}\,\frac{1 + \mathrm{Ai}'(\kappa^2)/\kappa\mathrm{Ai}(\kappa^2)}{1 - \mathrm{Ai}'(\kappa^2)/\kappa\mathrm{Ai}(\kappa^2)}\,, \tag{32}$$

and hence, after subtraction of the leading vacuum divergence,

$$\mathcal{E}_{\rm ren} = \frac{1-6\xi}{6\pi^2}\int_0^\infty d\kappa\,\kappa^3 e^{2\kappa z}\,\frac{1 + \mathrm{Ai}'(\kappa^2)/\kappa\mathrm{Ai}(\kappa^2)}{1 - \mathrm{Ai}'(\kappa^2)/\kappa\mathrm{Ai}(\kappa^2)}\,. \tag{33}$$

(Note that the energy vanishes if $\xi = \frac{1}{6}$.)

The integral for $\mathcal{E}_{\rm ren}$ can be computed without incident. It displays a weak divergence as $z \to 0^-$:

$$\mathcal{E} \sim -\frac{1}{192\pi^2}\frac{1}{z} \quad (\text{for } \xi = \tfrac{1}{4})\,. \tag{34}$$

It corresponds to a $z \ln z$ singularity in $\overline{T}$. This effect is attributable to diffraction off the singularity of the potential at $z = 0$; it goes away for larger α, as we will see.

Calculations have also been done inside the wall ($z, z' > 0$):

$$g_{\omega,\mathbf{k}_\perp}(z,z') = \pi \mathrm{Ai}(\kappa^2 + z_>)\mathrm{Bi}(\kappa^2 + z_<) - \frac{(\kappa\mathrm{Bi} - \mathrm{Bi}')(\kappa^2)}{(\kappa\mathrm{Ai} - \mathrm{Ai}')(\kappa^2)}\,\pi\mathrm{Ai}(\kappa^2 + z)\mathrm{Ai}(\kappa^2 + z')\,. \tag{35}$$

Before renormalization, with a temporary ultraviolet cutoff (arising naturally from point-splitting regularization), the energy density has the asymptotics

$$\mathcal{E} \sim \frac{3}{2\pi^2}\frac{1}{t^4} - \frac{z}{8\pi^2 t^2} + \frac{z^2}{32\pi^2}\ln t\,, \tag{36}$$

showing the expected "Weyl" terms correlating with the heat kernel expansion in presence of a potential $v(z) = z$ (but no wall). This formula displays two new divergences, but these "bulk" divergences are comparatively well understood in quantum field theory.

Removal of those terms has a physical interpretation. Include the dynamics of the v field in the theory, so that the total Lagrangian is (with $m = 0$)

$$\mathcal{L} = \frac{1}{2}\left[(\partial_t\varphi)^2 - (\nabla\varphi)^2 - m^2\varphi^2 - \varphi^2 v + (\partial_t v)^2 - (\nabla v)^2 - M^2 v^2 - Jv\right]. \tag{37}$$

The equations of motion are then

$$\Box\varphi = m^2\varphi + v\varphi\,, \qquad \Box v = M^2 v + \tfrac{1}{2}\varphi^2 + \tfrac{1}{2}J\,. \tag{38}$$

(The external source J is whatever it takes to support our static v.) The stress tensor can be defined and calculated by the Belinfante prescription; T_{00} acquires new terms proportional to M^2v^2 and Jv. Now recall (from heat-kernel theory) that the cut-off T_{00} contains terms of the types $t^{-2}v$, $\ln t\, v^2$, and $\ln t\, v''$. Thus $t^{-2}v$ and $\ln t\, v^2$ renormalize M and J. A v'' term in the action is formally a total divergence, so it does not contribute to the v equation of motion. (But it will not integrate to 0 in the total energy, since v has noncompact support.) When $\alpha = 1$ this last term is a delta function that does not show up in the Oklahoma calculation.

Remark. Instead of $\varphi^2 v$, the interaction $\varphi^2 v^2$ could be used. A detailed discussion of renormalization in that model, with a motivation very similar to ours, has been given very recently by Mazzitelli *et al.*[26]

Calculations can be done for a general α (and general ξ) via a WKB approximation for the reduced Green function. The solutions of

$$(-\partial_z^2 + \kappa^2 + z^\alpha)F^\pm(z) = 0 \tag{39}$$

are approximately

$$F^{\pm} \sim Q^{-1/4} \exp\left[\pm \int dz \left(Q^{1/2} + \frac{v''}{8Q^{3/2}}\right)\right], \tag{40}$$

with

$$Q \equiv \kappa^2 + v(z)\,, \quad \kappa = \sqrt{\mathbf{k}_\perp{}^2 - \omega^2}\,, \quad v(z) = z^\alpha \text{ (for } z > 0)\,. \tag{41}$$

From this input we find that *inside the wall*,

$$\mathcal{E} \approx \frac{3}{2\pi^2}\frac{1}{t^4} - \frac{v}{8\pi^2 t^2} + \frac{1}{32\pi^2}\left(v^2 + \tfrac{2}{3}(1-6\xi)v''\right)\ln t\,, \tag{42}$$

exhibiting the Weyl structure of divergences.

Outside (*but near*) *the wall*, we have

$$\mathcal{E}_{\mathrm{ren}}(z) \sim \frac{6\xi - 1}{96\pi^2}\,\Gamma(1+\alpha)\,|z|^{\alpha-2}\Gamma(2-\alpha, 2|z|)\,. \tag{43}$$

The singularity at $z = 0$ disappears for $\alpha > 2$:

$$\mathcal{E}_{\mathrm{ren}}(0) \approx \frac{1-6\xi}{96\pi^2}\,\frac{\Gamma(1+\alpha)2^{2-\alpha}}{2-\alpha}\,. \tag{44}$$

For $\alpha < 2$,

$$\begin{aligned}\mathcal{E}_{\mathrm{ren}}(z) &\sim \frac{6\xi - 1}{96\pi^2}\,\Gamma(1+\alpha)\left(|z|^{\alpha-2}\Gamma(2-\alpha) - \frac{2^{2-\alpha}}{2-\alpha}\right)\\ &\sim \frac{1-6\xi}{48\pi^2}\,(\gamma + \ln 2|z|) \quad \text{as } \alpha \uparrow 2\,.\end{aligned} \tag{45}$$

A numerical calculation of $\mathcal{E}_{\mathrm{ren}}$ was done in Ref. 25 for $\alpha = 2$, for which the exact $F^{\pm}$ are parabolic cylinder functions.

2.5. *Pressure*

The obvious next step is to calculate the other components of the stress tensor and then to verify that the energy-balance equation (14) is satisfied. This research is still in progress, but a few simple observations indicate that it is quite likely to succeed. In this preliminary skirmish we consider only the region outside the potential ($z < 0$), so that formulas (2) and (3) apply and the nontrivial renormalizations associated with v are not involved.

Consider first the terms independent of β. Because $\overline{T}$ in this problem is a function of $|\mathbf{r}_\perp - \mathbf{r}'_\perp|$, the three terms of that type in (3) collapse to $\frac{1}{2}\,\partial^2\overline{T}/\partial x_j{}^2$ for $j = 1$ or 2 (and we resume taking $\mathbf{r}'_\perp = 0$ without loss). Next, from (27) or indirectly from (30) one sees that $\overline{T}$ depends on t and x_j only through $s^2 = t^2 + |\mathbf{r}_\perp|^2$. A short calculation then shows that $\partial^2\overline{T}/\partial t^2$ and $\partial^2\overline{T}/\partial x_j{}^2$ are equal and opposite, apart from terms that vanish when all cutoffs are removed (in particular, $t = 0$, $x_j = 0$). Therefore, $\mathcal{E} + p_j = 0$ when $\beta = 0$.

For the terms proportional to β, the calculation is a bit more involved but the conclusion is simpler: The equation $\mathcal{E} + p_j = 0$ ($j = 1, 2$) holds for those terms by virtue of the equation of motion ($\Box\,\varphi = 0$), even when the points are separated, just as for the hard wall.

Thus the integrands in (14) are pointwise equal, as they were for the hard wall (15). The violation of the principle of virtual work in the theory of the hard wall is entirely a consequence of direction-dependent terms introduced by the point separation. The point of the soft-wall model is that no such cutoff should be necessary at the end of the calculation. The remaining issue (besides treating the inside of the potential) is whether the energy density and pressure in this model are indeed integrable when the points are no longer separated. As we have seen, this is false in the simplest case, $\alpha = 1$, because of a new (relatively mild) divergence inadvertently introduced by the nonsmoothness of the potential at $z = 0$, but it seems to be true for all larger alpha, certainly for $\alpha > 2$. Detailed calculations are ongoing.

3. Conclusions

- Understanding local energy density and pressure is essential for general relativity and clarifies the physics of global energy and force calculations.
- For hard (Dirichlet) walls, an ultraviolet cutoff yields physically inconsistent results for energy and pressure.
- Modifying the cutoff to point separation in a "neutral" direction yields physically plausible results, but logical justification is lacking.
- We seek to model a wall by a soft but rapidly increasing potential barrier, such as the power wall.
- Outside the potential, the effect of the soft wall is parametrized by the scattering phase shift, $\delta(p)$, whose asymptotics can be calculated at low and high frequency.
- We have "exact" formulas for $\langle T^{\mu\nu}\rangle$ in terms of the phase shift, but evaluating them is numerically challenging.
- Reorganizing the power-wall calculation gives rapidly converging integrals in terms of the eigenfunctions. Computations have been extended to the "inside" of the wall ($0 < z$).
- "Bulk" divergences inside the wall renormalize the equation of motion of the potential itself.
- Numerical computations have been done for $\alpha = 1$ (linear potential), but there are analytical results for general α. (Taking $\alpha \to \infty$ best approximates a hard wall (at $z = 1$).)
- The calculations are easily extended to general ξ. As usual, conformal coupling ($\xi = \frac{1}{6}$) yields the least singular results.
- Preliminary calculations of the pressures indicate that the expected energy balance (principle of virtual work) holds when α is sufficiently large to make the integrals for total energy and force converge.

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