

Wedges, Cosmic Strings, and the Reality of Vacuum Energy

Stephen A. Fulling et al.

Texas A&M University

NATIONAL SCIENCE FOUNDATION GRANT PHY-0554849

ET AL.

Undergraduate students:

Phuongmai (“Mai”) Truong
(now at U.C. – Berkeley)

Cynthia Trendafilova

Graduate students:

Jeffrey (“Jeff”) Bouas
Fernando Mera

Postdoc:

Jeffrey (“Jef”) Wagner

Special thanks to Jef Wagner for most of the work on the *Mathematica* graphics.

My junior colleagues are not responsible for any polemical or philosophical remarks by me that someone may find objectionable.

We consider

- Massless scalar field (but in 3 space dimensions).
- Dirichlet boundary conditions (ignoring the possibility of a Zarembo–Seeley phenomenon at the cosmic string).
- Exponential ultraviolet cutoff when needed.
(t = “imaginary time” = cutoff parameter.)
- Start with “elementary” curvature coupling $\xi = \frac{1}{4}$, then add corrections for conformal ($\xi = \frac{1}{6}$) or minimal ($\xi = 0$) coupling.

Some old Casimir theory

Vacuum energy can be calculated from a Green function (such as the cylinder kernel).

$$T_{00} = -\frac{1}{2} \frac{\partial^2 \overline{T}}{\partial t^2} + \left(\xi - \frac{1}{4} \right) \nabla_{\mathbf{r}} \cdot [\nabla_{\mathbf{r}'} \overline{T}]_{\mathbf{r}'=\mathbf{r}} .$$

$$\overline{T}(t, \mathbf{r}, \mathbf{r}') = - \sum_{n=1}^{\infty} \frac{1}{\omega_n} \phi_n(\mathbf{r}) \phi_n(\mathbf{r}')^* e^{-t\omega_n} .$$

Free space: $G_0(\mathbf{r}, t; \mathbf{r}', t') = \frac{-1/2\pi^2}{(t - t')^2 + \|\mathbf{r} - \mathbf{r}'\|^2} .$

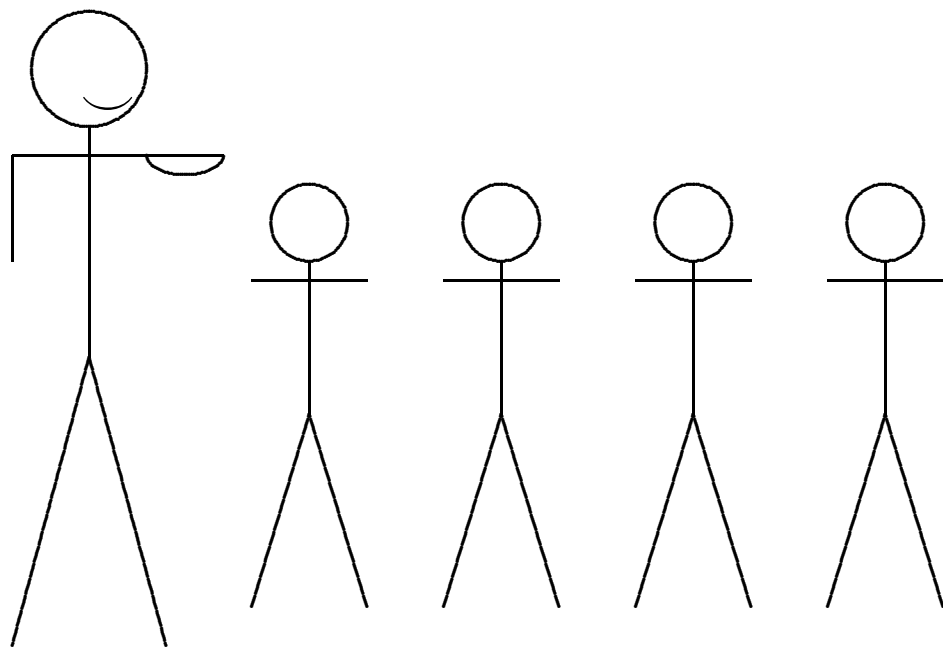
Plane boundary: Method of images. (Suppress y, z .)

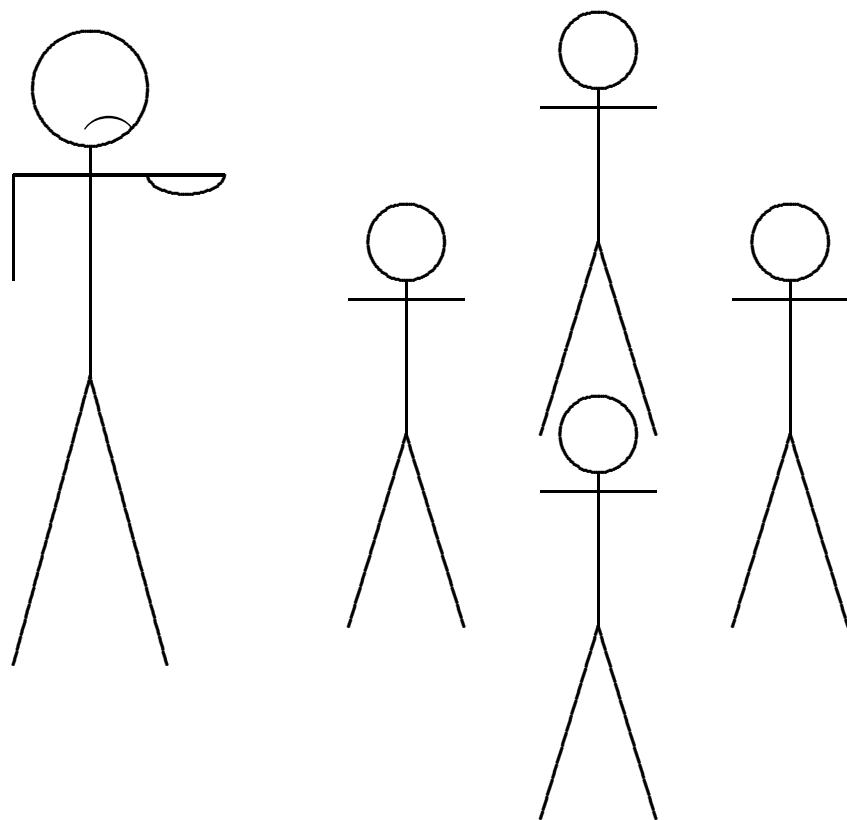
$$G(x, t; x', t') = G_0(x, t; x', t') - G_0(x, t; -x', t').$$

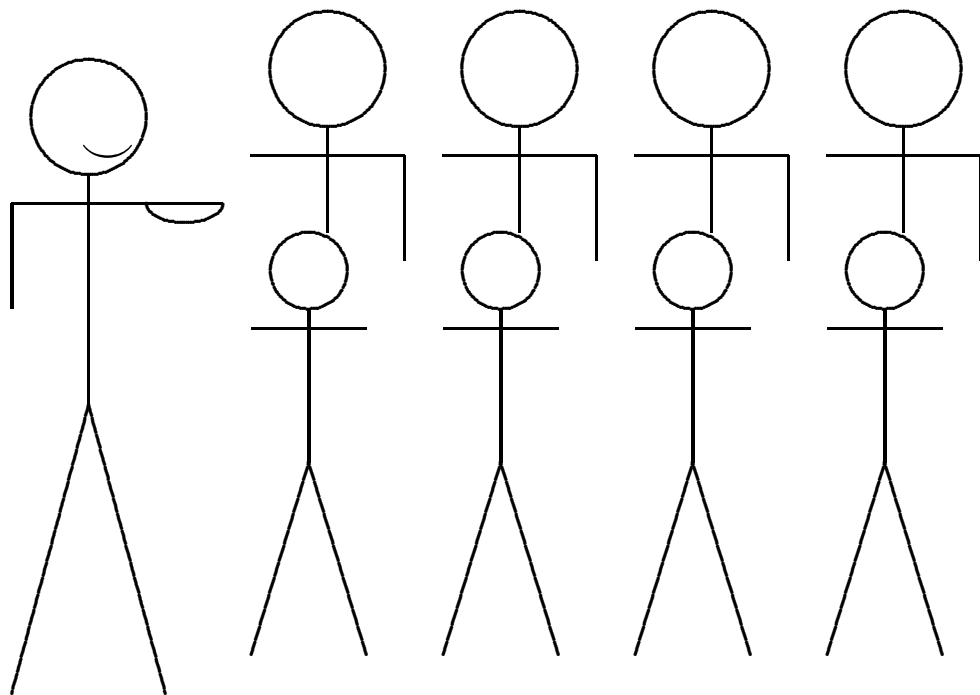
$$\begin{array}{ccc} - & | & + \\ \mathbf{r}' & & \mathbf{r} \end{array}$$

THE CRITIQUE

- A real physicist is interested in the scalar field only as a model of the E&M field.
- The E&M Casimir energy is energy of interaction of fluctuations of the electrons in the conductors. The field energy is just a bookkeeping device.
- **In the periodic universe there are no boundaries, no fluctuating electrons. There is no evidence that cosmological vacuum energy exists. (Dark energy is too small by 10^{-120} .)**

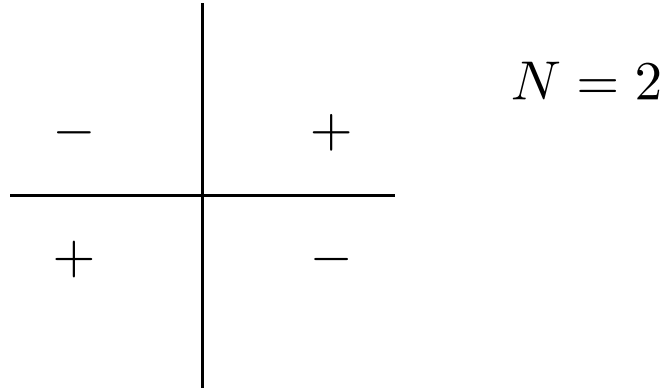






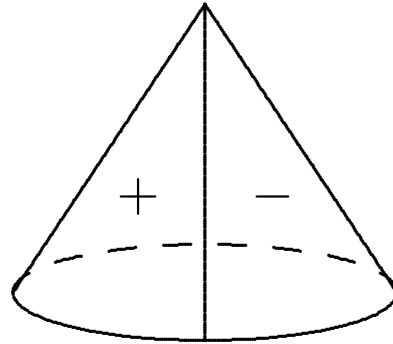
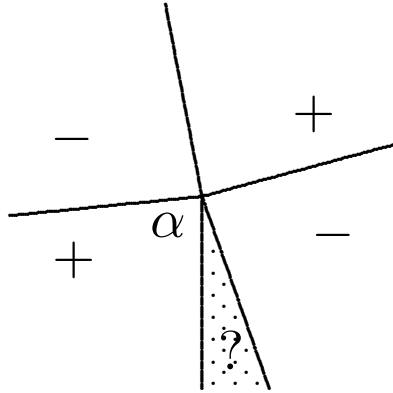
Wedges and cones

WEDGES WITH NICE ANGLES



If $\alpha = \frac{2\pi}{2N}$, the cylinder kernel (or other Green function) can be found by the classic method of images.

WEDGES WITH BAD (I.E., GENERIC) ANGLES



$2N$ copies of the wedge do not fit into the plane, but do fit into a cone of defect angle $2\pi - 2N\alpha$ (which may be negative: $0 < \alpha < \infty$).

CONES (INFINITELY THIN COSMIC STRINGS)

So, what is the Green function on a cone? It must be periodic with period $\theta_1 = 2\pi - \text{defect angle} = 2N\alpha$. There is no reason not to choose $N = 1$, so $\theta_1 = 2\alpha$ in the polar problems just as $L_1 = 2L_0$ in the Cartesian problems.

The analog of the free Green function G_0 is the one for the cone with angle $\theta_1 = \infty$, an infinite-sheeted Riemann surface. (z, t suppressed.)

$$G_{\theta_1}(\theta, \theta') = G_{\infty}(\theta, \theta') + G_{\infty}(\theta, \theta' + \theta_1) + \cdots$$

Note that $G_{2\pi}$ is now the original free-space Green function in polar coordinates.

This infinite-sheeted cone is the *Dowker manifold*:

J. S. Dowker, *J. Phys. A* **10** (1977) 115, and later papers (channeling Sommerfeld and Carslaw on diffraction theory)

Historically, cone manifolds (Riemann surfaces) were introduced to study wedges by the method of images. More recently cosmic strings were studied by analogy with wedges, though the former are more elementary.

If we can calculate vacuum energy around cosmic strings, we can do it around wedges (though the understanding of divergences is more problematical in the latter case).



Vacuum energy around a cosmic string renormalizes the mass/length of the string. But there is no string at the vertex of a conducting wedge.

Polar–Cartesian comparison

In the polar plane,

$$G_{2\pi}(\theta, \theta') = \sum_{n=-\infty}^{\infty} G_{\infty}(\theta, \theta' + 2\pi n).$$

In the periodic universe,

$$G_{L_1}(x, x') = \sum_{n=-\infty}^{\infty} G_0(x, x' + nL_1).$$

$G_{2\pi}$ and G_0 are the free-space Green function.

In terms of normal modes, $G_{2\pi}$ is a sum over angular momentum quantum number, G_∞ an integral over it. G_{L_1} is a Fourier sum, G_0 a Fourier integral.

$G_{2\pi}$ and G_0 are the free-space Green function that gives the zero-point energy density that must be subtracted from that of any other configuration.

If you believe in the stress tensor of quantum field theory, all this is totally consistent and unsurprising.

But if you don't, you are forced into an untenable position:

- To calculate vacuum energy in the periodic universe, you must ignore the (mathematically appropriate) Fourier sum in favor of the integral (since you don't believe vacuum energy can exist in the absence of van der Waals sources).
- In polar coordinates, you must use the sum to get the right answer for empty Euclidean space. The integral gives something else, the energy density of the Dowker manifold.

I see no possibility of a theoretical justification for this ad hoc switch of point of view.

Green function and vacuum energy on cones

PARTIAL BIBLIOGRAPHY (besides Dowker)

- W. Lukosz, *Z. Phys.* **262** (1973) 327.
- A. G. Smith, in *The Formation and Evolution of Cosmic Strings*, ed. G. Gibbons et al. (C.U.P., 1990) 263.
- M. E. X. Guimarães and B. Linet, *Commun. Math. Phys.* **165** (1994) 297.

u , THE KEY INGREDIENT

Recall that Green functions in their full glory depend on variables $(r, \theta, z, t; r', \theta', z', t')$ (where t could also be x^0 , ω , or s).

We can set $t', z', \theta' = 0$. Define u by

$$2rr' \cosh u = r^2 + r'^2 + z^2 + t^2$$

or

$$u = -\ln \frac{r_2 - r_1}{r_2 + r_1} \quad \text{where}$$

$$r_1 = \sqrt{(r - r')^2 + z^2 + t^2}, \quad r_2 = \sqrt{(r + r')^2 + z^2 + t^2}.$$

THE CYLINDER KERNEL

$$G_{\infty} = - \frac{1}{2\pi^2 r r' \sinh u} \frac{u}{u^2 + \theta^2} .$$

$$G_{\theta_1} = - \frac{1}{2\pi\theta_1 r r' \sinh u} \frac{\sinh\left(\frac{2\pi u}{\theta_1}\right)}{\cosh\left(\frac{2\pi u}{\theta_1}\right) - \cos\left(\frac{2\pi\theta}{\theta_1}\right)} .$$

$$G_{2\pi} = - \frac{1}{4\pi^2 r r'} \frac{1}{\cosh u - \cos \theta} = \frac{-1/2\pi^2}{t^2 + \|\mathbf{r} - \mathbf{r}'\|^2} .$$

Note that $G_\infty(\theta, \dots)$ looks suspiciously like $G_{2\pi}(x - x', \dots)$.

Correspondingly, G_{θ_1} has a structure similar to the G_{L_1} for the periodic universe.

In summary, G_∞ is found by separation of variables (sum over modes). G_{θ_1} can be found likewise, but also can be found from G_∞ by images (sum over paths). (G_∞ tells how to diffract a path off the isolated conical singularity.) These remarks extend to the energy density, and to wedges.

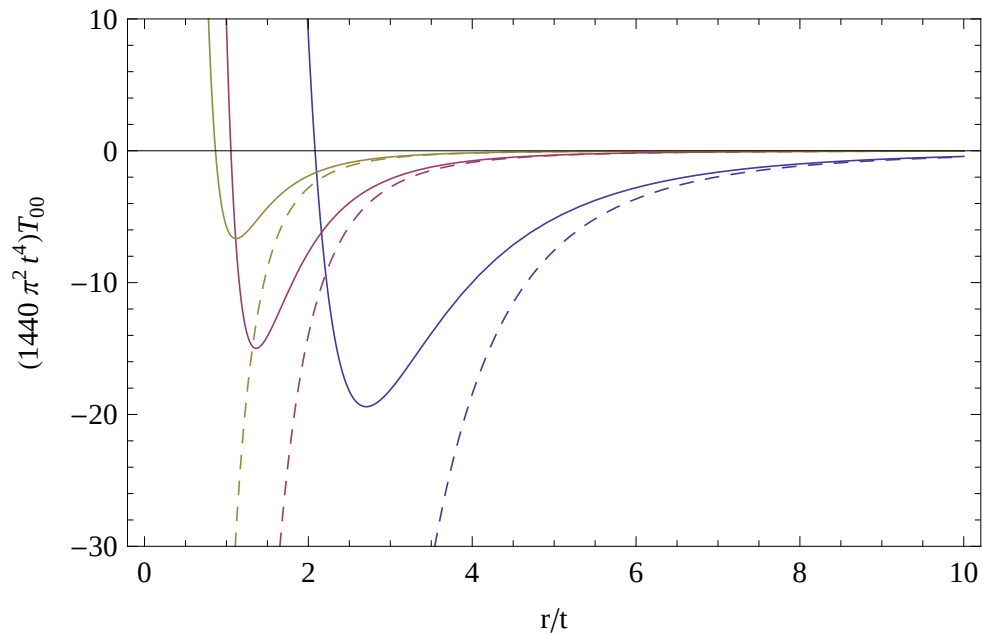
THE ENERGY DENSITY

With curvature coupling $\xi = \frac{1}{4}$, we need only t derivatives of G , with $r = r'$, $z = 0$, and $\theta - \theta' = 0$ (“on diagonal”). (Cutoff t taken small but nonzero.)

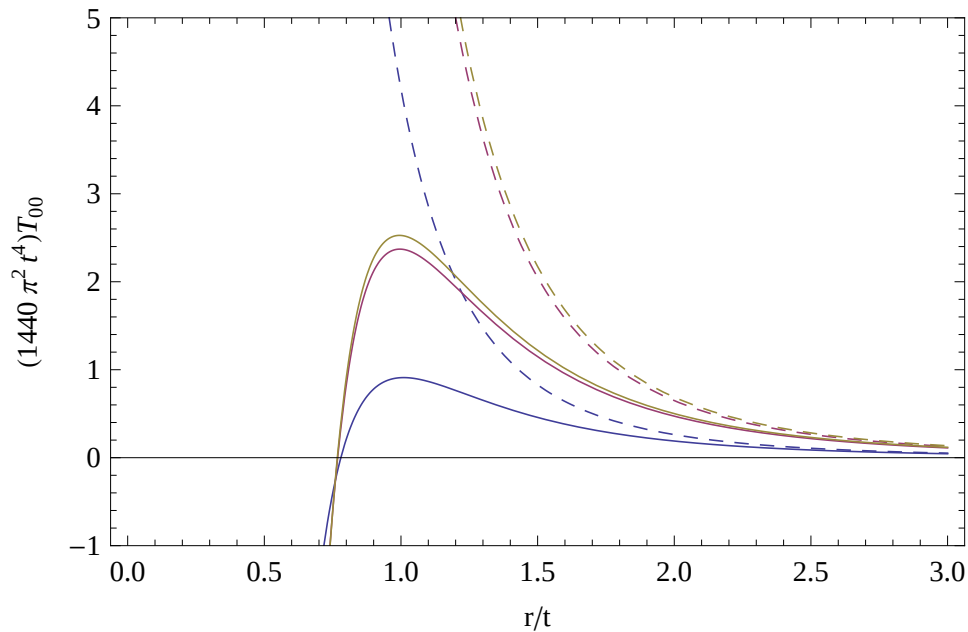
$$T_{00}(r, t) = -\frac{1}{2} \frac{\partial^2 G}{\partial t^2} = \frac{1}{t^4} F\left(\frac{r}{t}\right).$$

We have formulas, but they are complicated and uninformative. *Mathematica* plotting proves indispensable.

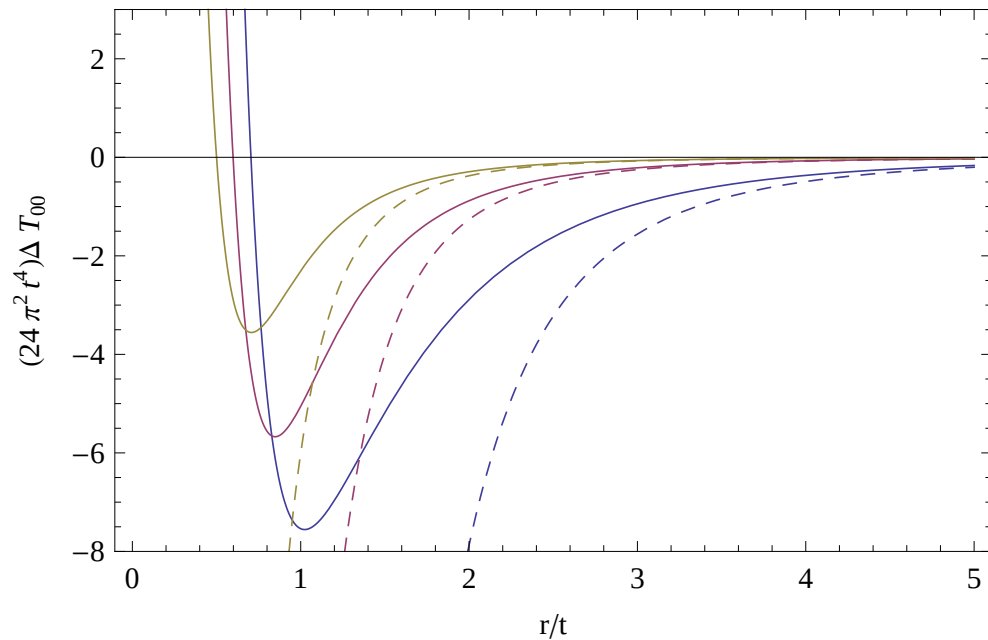
ENERGY DENSITY ($\xi = \frac{1}{4}$) FOR CONE ANGLES $\frac{\pi}{4}$, $\frac{3\pi}{5}$, π



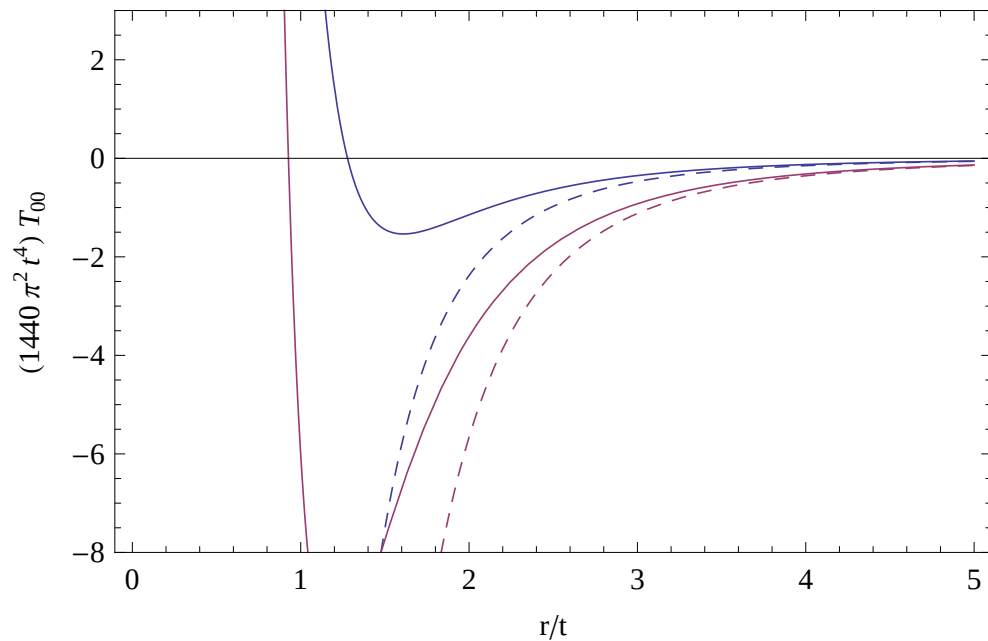
ENERGY DENSITY ($\xi = \frac{1}{4}$) FOR CONE ANGLES $\frac{5\pi}{2}, 8\pi, \infty$



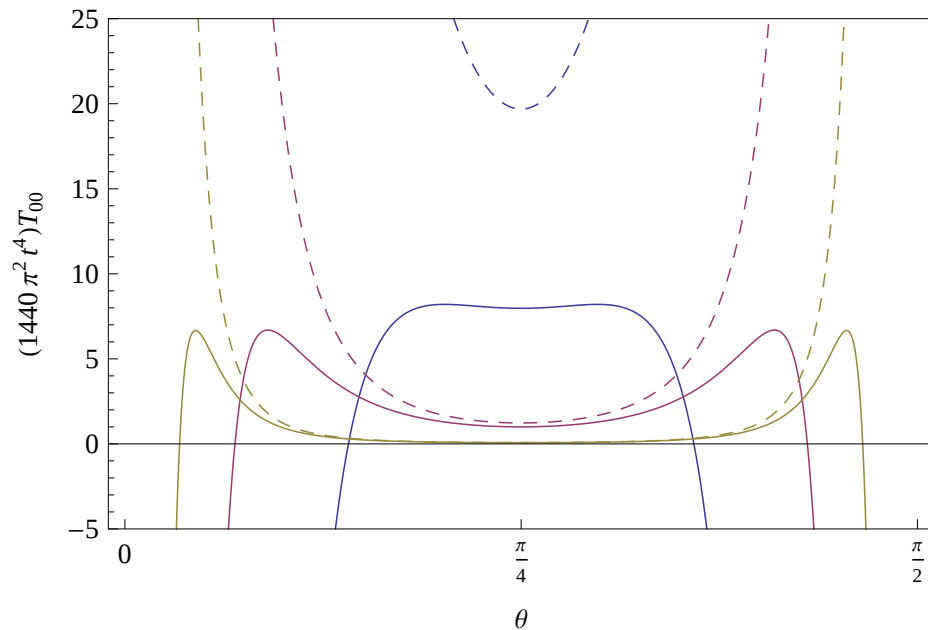
CURVATURE COUPLING TERM FOR CONE ANGLES $\frac{\pi}{4}$, $\frac{3\pi}{5}$, π



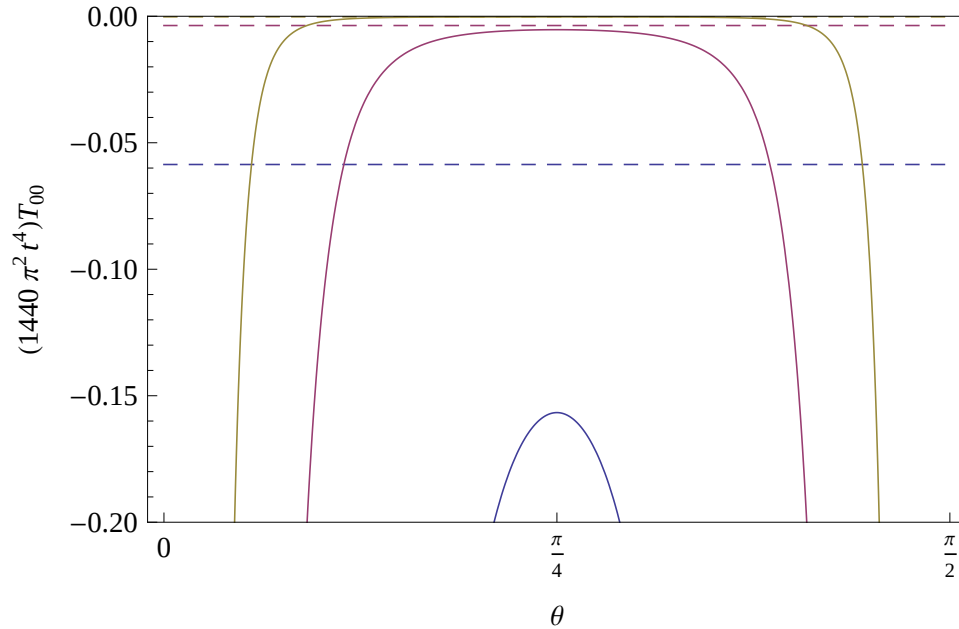
ENERGY DENSITY FOR CONE ANGLE $\frac{4\pi}{5}$ AND $\xi = \frac{1}{4}, \frac{1}{6}$



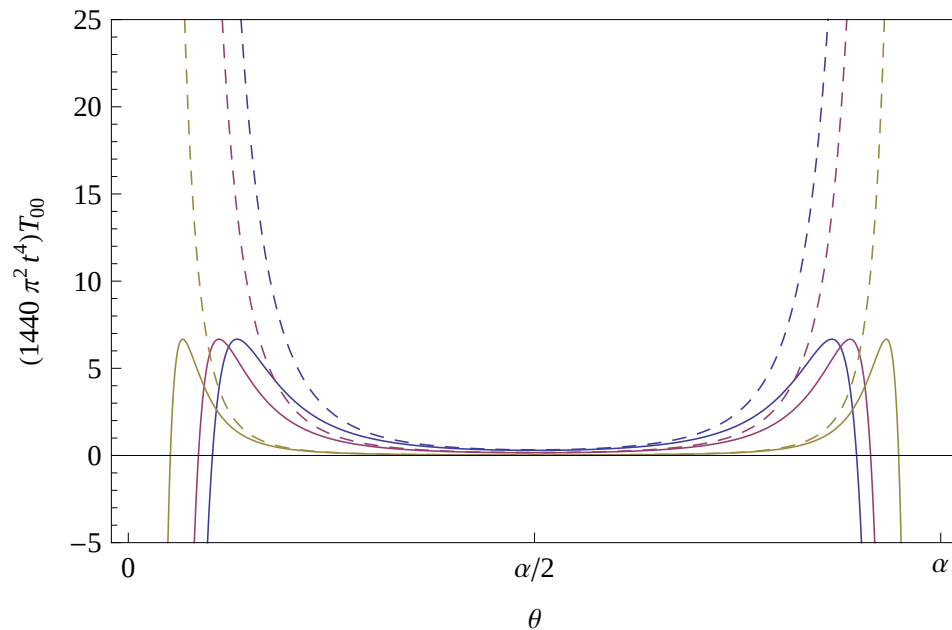
WEDGE ENERGY DENSITY ($\xi = \frac{1}{4}$) FOR $r = 2, 4, 8$



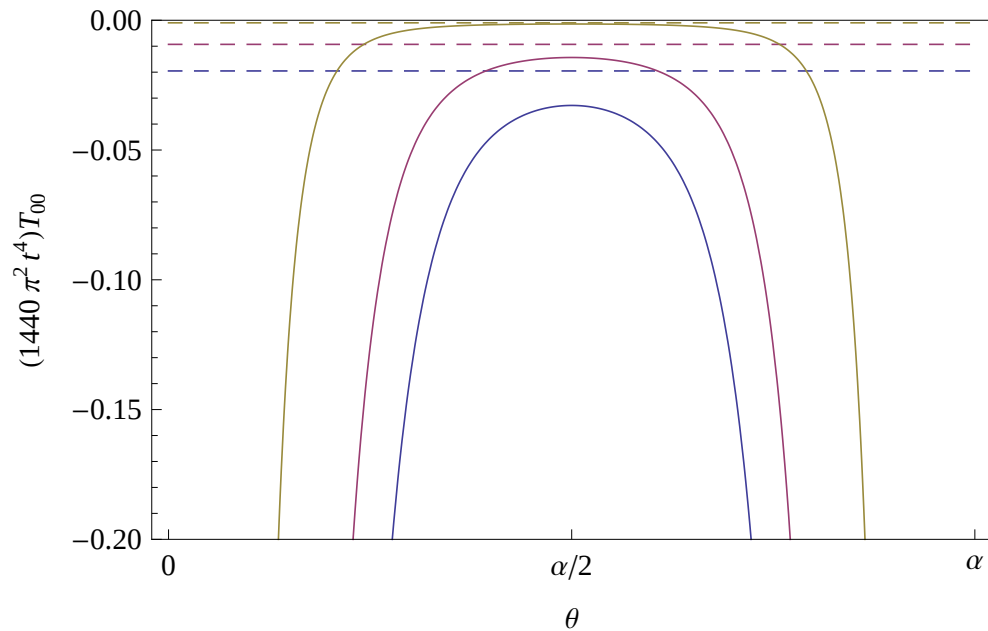
WEDGE ENERGY DENSITY (CONFORMAL) FOR $r = 4, 8, 16$



WEDGE ANGLES $\frac{\pi}{3}, \frac{2\pi}{5}, \frac{2\pi}{3}; r = 8, \xi = \frac{1}{4}$



WEDGE ANGLES $\frac{\pi}{3}, \frac{2\pi}{5}, \frac{2\pi}{3}; r = 8, \xi = \frac{1}{6}$



ENERGY DENSITY ($\xi = \frac{1}{4}$) FOR WEDGE ANGLE $\frac{2\pi}{3}$

and

ENERGY DENSITY (CONFORMAL) FOR WEDGE ANGLE $\frac{2\pi}{5}$

In the limit of no cutoff, the conformal energy density is independent of the angle coordinate:

$$T_{00} = \frac{1}{1440r^4\alpha^2} \left(\frac{\pi^2}{\alpha^2} - \frac{\alpha^2}{\pi^2} \right)$$

for a wedge of angle α . (D. Deutsch and P. Candelas, *Phys. Rev. D* **20** (1979) 3063)

Summary

1. Wedge $\Leftarrow$ cone + image(s).
2. Cone $\Leftarrow$ Dowker + periodic.
3. Plane and cone functions qualitatively similar (except for conformal coupling).
4. Wedge effects = plane + cone.
5. Flat space = cone of angle 2π . Discreteness of angular modes cancels vacuum energy of Dowker space.

6. We calculate **local energy density** for **all values** of the curvature coupling constant ξ .
7. Conformal coupling ($\xi = \frac{1}{6}$) removes plane and wedge-side divergences
($\Rightarrow$ flat function of angle in limit of no cutoff)
but not cone and wedge-vertex divergences.
8. Total energy (per length) is independent of ξ **only** when cutoff is retained.
9. For any given θ_1 there is a (rather large) ξ for which the r^{-4} leading vertex divergence vanishes.