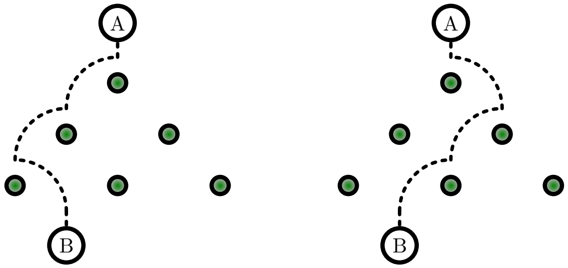
Universality in chaotic quantum transport: the concordance between random matrix and semiclassical theories

Gregory Berkolaiko, Texas A&M University

Based on work with J.Kuipers

IMS, Singapore, June 18, 2012

A handwaving introduction to Quantum Mechanics



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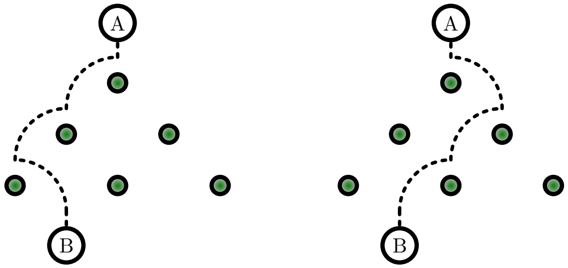

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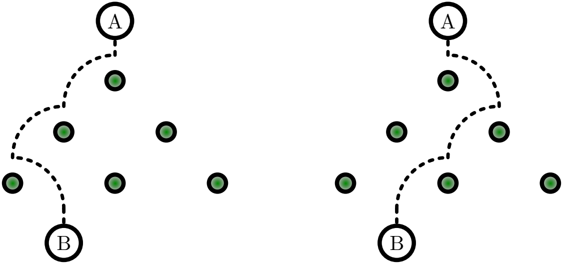
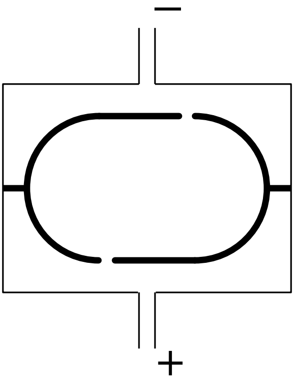


- **Probability**: $P(A \to B) = P_{path1} + P_{path2}$,
- **QM**: $\Psi(A \to B) = \Psi_{path1} + \Psi_{path2}$, where $|\Psi_{path}|^2 = P_{path}$.
- But Ψ are not real positive, so interference is possible (due to phases).

Quantum Transport: an experimental setup



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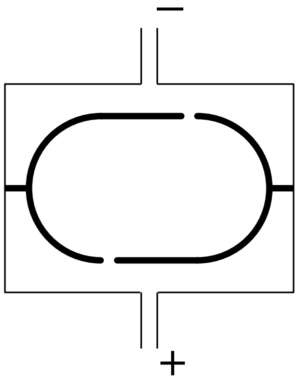
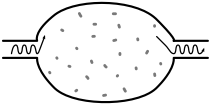


- Prepare sample: cut through conducting layer to make a cavity with small openings
- Apply voltage and measure current fluctuations

Quantum Transport: theory (one channel per lead)



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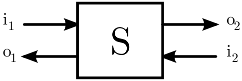

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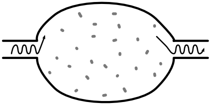

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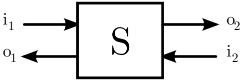

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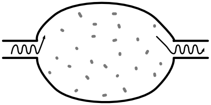

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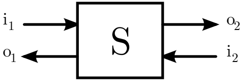

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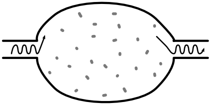

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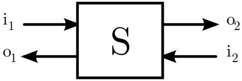

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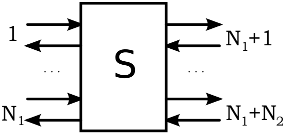
- Shot noise intensity is (variance of the binomial process)

$$\langle \delta I^2 \rangle = 2e^2\frac{eV}{h}T(1-T).$$

Quantum Transport: theory (multiple channels)

Typically there are many channels (10^7 for a wire of radius 1 μm).



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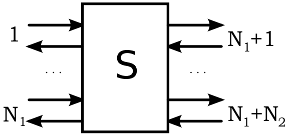

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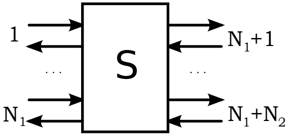

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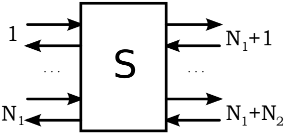

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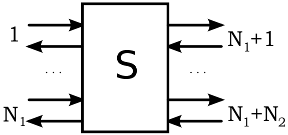

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- Shot noise intensity $\sim \text{tr}\left(t^* t - (t^* t)^2\right)$.

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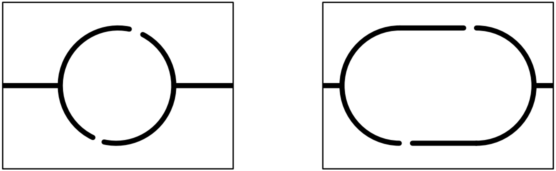
- If time-reversal (TR) invariant, then

$$\Psi(A \to B) = \Psi(B \to A) \Longrightarrow S = S^T.$$

- Verified by experiment (e.g. Chang, Baranger, Pfeiffer, West '94): properties of S depend on classical properties of the cavity shape:



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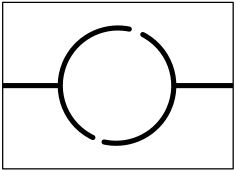

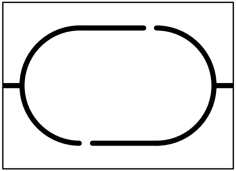

But how to access S ?

Random Matrix model

Blümel and Smilansky'88: take S to be a random matrix with suitable symmetry.

- Broken TR:
  - U unitary,
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- TR:
  - V unitary symmetric,
  - measure invariant under $V \mapsto W^T V W$ for any unitary W,
  - $V = U^T U$, where $U \in \mathrm{CUE}(N)$,
  - **Circular Orthogonal Ensemble (COE)**,
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U(N) integration

Example 1: Calculate conductance

$$\langle \mathrm{tr}(t^* t) \rangle = \left\langle \sum_{i=1}^{N_1} \sum_{o=N_1+1}^{N} U_{oi} \overline{U_{oi}} \right\rangle_{U(N)} = N_1 N_2 \left\langle U_{oi} \overline{U_{oi}} \right\rangle_{U(N)}$$

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Theorem (Samuel'80)

$$\langle U_{a_1 b_1} \cdot \cdot U_{a_s b_s} \overline{U_{\alpha_1 \beta_1}} \cdot \cdot \overline{U_{\alpha_t \beta_t}} \rangle_N = \delta_t^s \sum_{\sigma, \pi \in S_t} V_N(\sigma^{-1}\pi) \prod_{k=1}^{t} \delta_{a_k}^{\alpha_{\sigma(k)}} \delta_{b_k}^{\beta_{\pi(k)}}.$$

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Example 2: For $\left\langle U_{12} U_{13} \overline{U_{13}}\, \overline{U_{12}} \right\rangle$ use $\pi = (12)$ and $\sigma = \mathrm{id}$ or (12).

U(N) integration (cont)

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where V_N *are class functions that satisfy* $V_N(\emptyset) = 1$ *and*

$$NV(c_1, \ldots, c_k) + \sum_{p+q=c_1} V(p, q, c_2, \ldots, c_k) + \sum_{j=2}^{k} c_j V(c_1 + c_j, \ldots, \hat{c}_j, \ldots, c_k) = \delta_{c_1,1} V(c_2, \ldots, c_k),$$

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- $V(1,1) = 1/(N^2-1)$ and $V(2) = -1/(N^2-1)N$.
- Sometimes V_N is called "Weingarten function".

Further RMT references

- Review: Beenakker'97
- First few moments: Baranger, Mello 94; Jalabert, Pichard, Beenakker '94
- Diagrammatics: Brouwer and Beenakker '94
- From Selberg integrals: Savin, Sommers, Wieczorek '06-'08; P. Vivo, E. Vivo '08; Novaes '08; Livan, P.Vivo '11; Mezzadri, Sims '11.

Semiclassical approximation

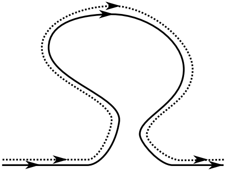
RM model is very successful but why? Limits of applicability?
Semiclassical approximation

$$S_{oi}(E) \sim \sum_{\gamma(i\to o)} A_\gamma(E) e^{\frac{i}{\hbar} S_\gamma(E)},$$

so, for example,

$$\langle S_{oi}\overline{S_{oi}}\rangle \sim \left\langle \sum_{\gamma,\gamma'} A_\gamma \overline{A_{\gamma'}} e^{\frac{i}{\hbar}(S_\gamma(E)-S_{\gamma'}(E))} \right\rangle_E ,$$

$S_\gamma \approx S_{\gamma'} \quad \Longrightarrow \quad \gamma'$ must mostly follow γ.



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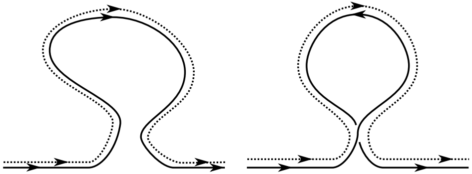

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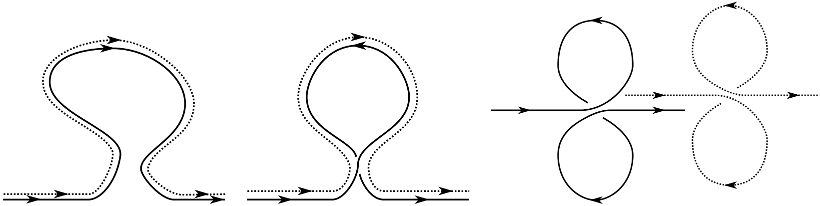

Higher order correlations

A typical term in $\langle \mathrm{tr}(t^* t)^2 \rangle$ is

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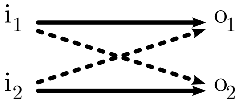
that is

$$\begin{array}{ll} \gamma_1 : i_1 \longrightarrow o_1 & \gamma_2 : i_2 \longrightarrow o_2 \\ \gamma_1' : i_2 \longrightarrow o_1 & \gamma_2' : i_1 \longrightarrow o_2. \end{array}$$



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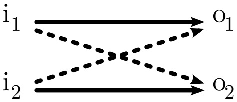
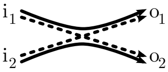


Examples:



Higher order correlations

A typical term in $\langle \mathrm{tr}(t^* t)^2 \rangle$ is

$$\langle S_{o_1 i_1} S_{o_2 i_2} \overline{S_{o_1 i_2} S_{o_2 i_1}} \rangle$$

that is

$$\gamma_1 : i_1 \to o_1 \qquad \gamma_2 : i_2 \to o_2$$
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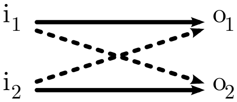

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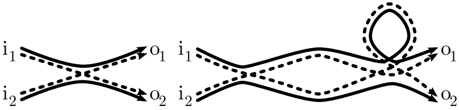

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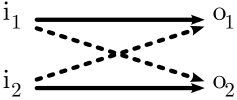

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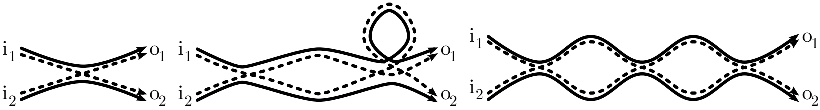
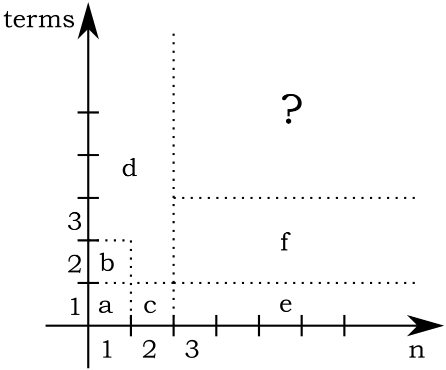


Transport moments in the universal regime — history

Task: Evaluate $\mathrm{tr}\left[(t^*t)^n\right]$ to all orders in $1/N$.

(Universal regime only: $\tau_E \ll \tau_d$)



(a) Diagonal approximmation: Blümel and Smilansky '88

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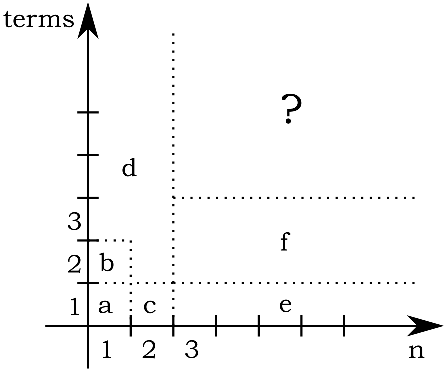

(b) Breakthrough of Richter and Sieber '02: first off-diagonal term

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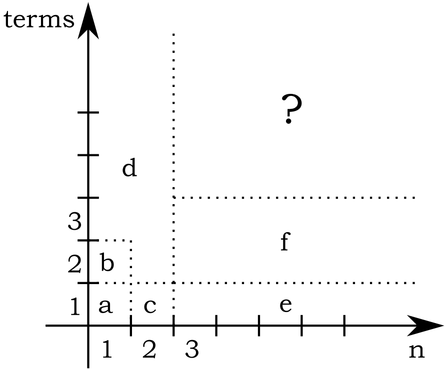

(c) First term for shot noise: Schanz, Puhlmann and Geisel '03

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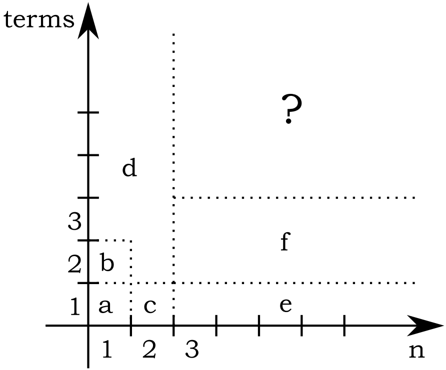

(d) $\operatorname{tr}(t^*t)$ and $\operatorname{tr}(t^*t)^2$ to all orders of $1/N$ (and diagram evaluation rules): Müller, Heusler, Braun and Haake '07

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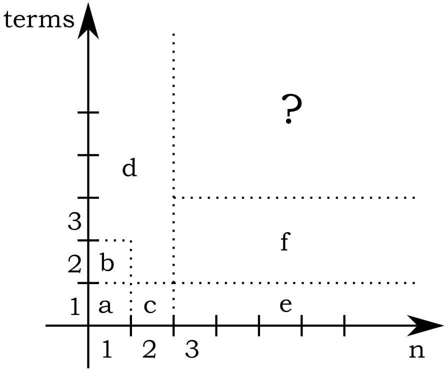

(e) first term of $\mathrm{tr}(t^*t)^n$ for all n: GB, Harrison and Novaes '09

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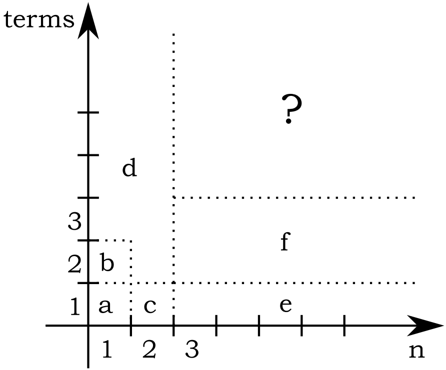


(f) first several terms of $\mathrm{tr}(t^*t)^n$ for all n: GB and Kuipers '11

Evaluation rules

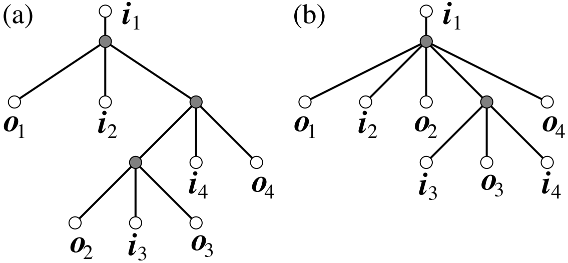
Müller, Heusler, Braun and Haake ’07:

- Each vertex: $\times(-N)$
- Each edge: $\times\frac{1}{N}$

Assumes chaotic system with fast equilibration and long stretches between encounters.
Uses Hannay - Ozorio de Almeida sume rule.

Leading order diagrams: combinatorial meaning

From leading order contributions to $\mathrm{tr}(t^* t)^4$:



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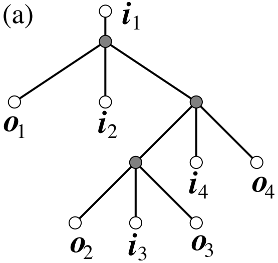

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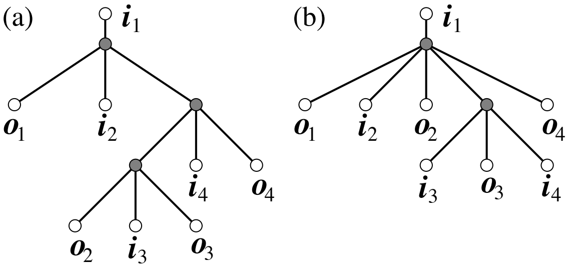

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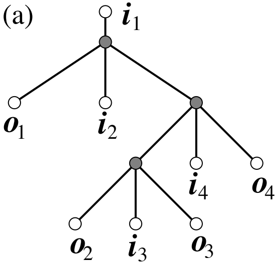
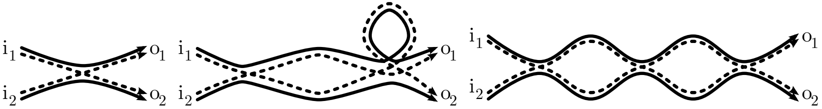


- All diagrams are *plane* trees.
- Correspond to factorizations of the permutation $(1\,2\,3\,4)$, e.g. $(1\,2)(2\,4)(2\,3)$ and $(3\,4)(1\,2\,4)$.
- Factorizations count up to the permutation of commuting factors, e.g. $(1\,3)(1\,2)(3\,4) = (1\,3)(3\,4)(1\,2)$

Mathematical description of diagrams

Maps or **ribbon graphs**: graphs embedded on a surface.
Examples:



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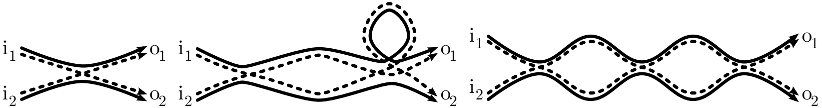

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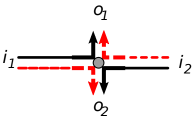

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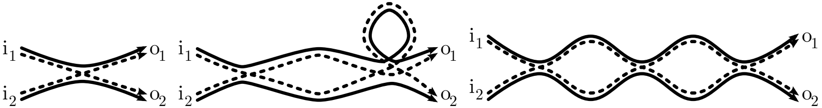

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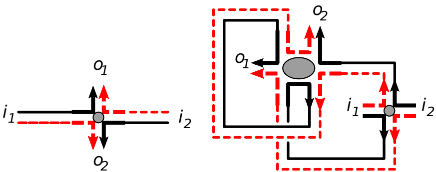

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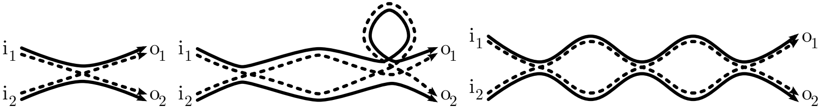

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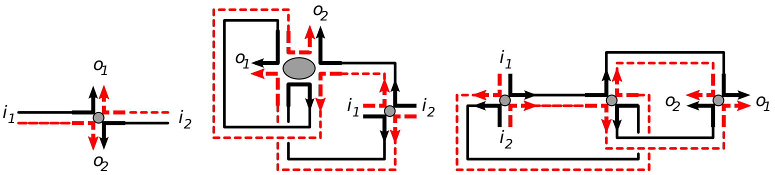
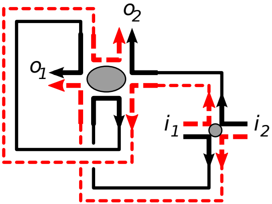


Conditions on diagrams

To be a valid diagram for $\mathrm{tr}(t^* t)^n$,



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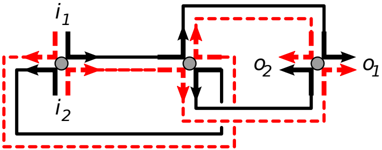

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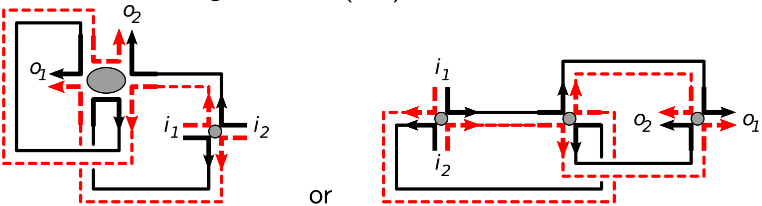

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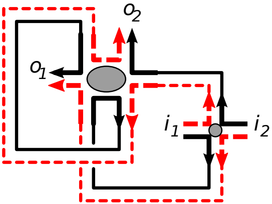

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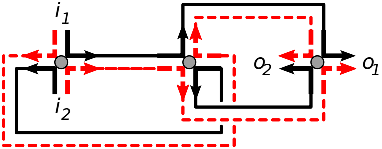

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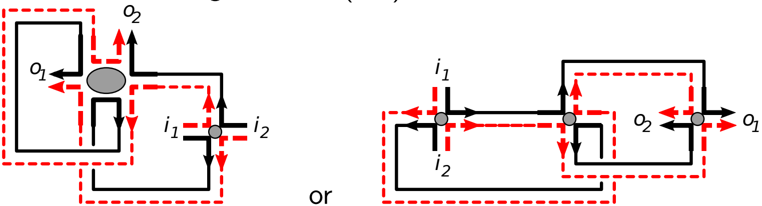

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- Each edge is traversed twice, by red and by black $\Longrightarrow$ each cycle has an even number of odd vertices.

Factorizations and diagrams

(B. and Irving '08-15) The inequivalent factorizations (of $\pi = (1\,2\ldots n)$) are enumerated by maps that have

- $2n$ vertices of deg 1: $i_1, \ldots, i_n, o_1, \ldots, o_n$
- all other vertices degree $2m \geq 4$
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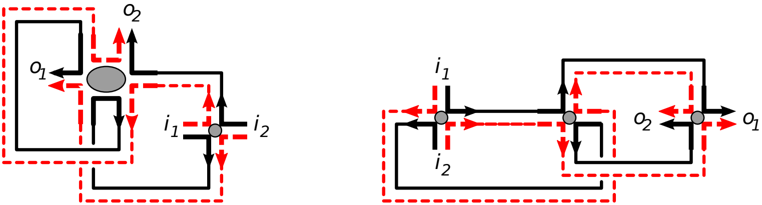
This is a subset of semiclassical diagrams!

Contribution of a diagram



Each diagram contributes (Müller, Heusler, Braun and Haake '07):

$$(-N)^V \frac{1}{N^E} = (-1)^V \frac{N}{N^{2g}},$$

where genus g is

$$2g = 1 - V + E - F + C$$

Number of diagrams $D_{2g}(\pi, V)$ (in all examples $\pi = (1\,2\,\ldots\,n)$).

Notation

Integer partitions, for example

$$11 = 1 + 1 + 1 + 2 + 3 + 3$$

For partition α of n,

- $\alpha = [1^{\alpha_1} 2^{\alpha_2} 3^{\alpha_3} \cdots]$,
- **length** $|\alpha| = |\alpha_1| + |\alpha_2| + |\alpha_3| + \ldots$
- **depth** $\langle\alpha\rangle = 0 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + \ldots$.

For the example above, $\alpha = [1^3 2^1 3^2]$ with length 6 and depth 5.

Factorizations

$$\pi = \sigma_1 \sigma_2 \cdots \sigma_k, \qquad \pi \in S_n$$

- **Ordered factorizations:** $\sigma_1, \ldots, \sigma_k \in S_n$.
  - Example: $(12345) = \Big[(134)\Big]\Big[(12)(45)\Big]$.

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- **Primitive factorizations:** $\sigma_1, \ldots, \sigma_k$ are transpositions.
  - $\sigma_k = (s_k\, t_k)$, with $s_k < t_k$ and $t_{k-1} \leq t_k$.
  - Example: $(12345) = (12)(45)(35)(25)$.

  Number denoted by $p_d(\pi)$, where $d = k$.

Factorizations, RMT and diagrams

Theorem

The CUE coefficients

$$V(\pi) = \frac{(-1)^{\langle \pi \rangle}}{N^n} \sum_{d=1}^{\infty} w_d(\pi) \frac{(-1)^d}{N^d},$$

where

$$w_d = \sum_k (-1)^{k-d} A[\pi, k, d] \qquad \text{(Collins'03)}$$

(GB-Irving)

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$$= \sum_{V:d=\langle\pi\rangle+2g} (-1)^V D_{2g}(\pi, V) \qquad \text{(GB-Kuipers)}$$

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Concordance between RMT and semiclassics

Corollary

Since the RMT agrees with semiclassics on the level of the class functions $V_N(\pi)$, they will also agree for all moments (including nonlinear).

Proof idea

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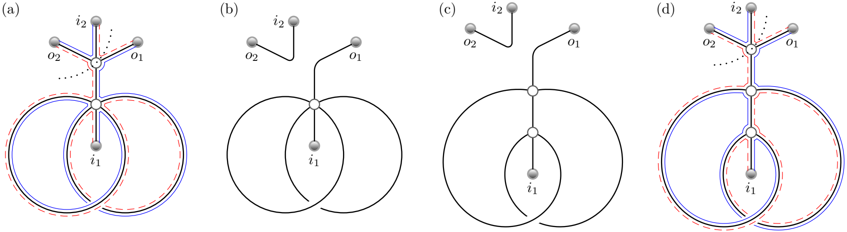

1. Untie o_{max} (while we can).
2. If cannot, insert / remove edge.
3. Tie things back.

If step 2 was performed we have two diagrams that cancel.
If not, we have a primitive factorization.

Summary

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  but it all works out.
- Semi-automatic way to calculate generating function for low-order (up to 7) corrections to $\mathrm{tr}(t^* t)^n$.