Birman - Schwinger Principle

Count <0 eigenvalues of $-\Delta - V$, $V>0$

$$(-\Delta - V)\,u = -\lambda u$$

$$(-\Delta + \lambda)\,u = Vu \qquad V^{1/2}u = v.$$

$$(-\Delta + \lambda)\,V^{-1/2}v = V^{1/2}v$$

$$v = \underbrace{V^{1/2}(-\Delta+\lambda)^{-1}V^{1/2}}_{B(\lambda)}\,v$$

$-\lambda$ - eig of $-\Delta - V$ $\iff$ $B(\lambda)$ has eig 1

$\lambda \searrow$ $\qquad (-\Delta+\lambda)^{-1} \nearrow$ $\qquad B(\lambda)\nearrow$

Take λ from $+\infty$ to λ_0

$$\therefore \quad \#\{\text{eig} < -\lambda_0 \text{ of } -\Delta - V\} = \#\{\text{eig} > 1 \text{ of } B(\lambda_0)\}.$$

"Birman - Schwinger Principle" B '61
S '61

Simon "Trace Ideals", Gesztesy - Makarov '00 "gaps"
Pushnitsky '11 "On ess spec

Birman - Schwinger on discrete spec (matrices)

H - Hermitian on $\mathbb{R}^n$ $\qquad \Omega = \begin{pmatrix} I_{\omega^-} & \\ & I_{\omega^+} \end{pmatrix}$ on $\mathbb{R}^n$

$K : \mathbb{R}^n \to \mathbb{R}^m$

$H = A + K^* \Omega K$ pert. of rank m (if K row indep.)

<u>Thm</u>: (Ges-Mak, GB-Kuch) The Morse index (# neg. eig.) of

$$B(\lambda) = \Omega^{-1} - K (H - \lambda)^{-1} K^*$$

is equal to $\omega_- + \text{shift}(\lambda; A, H)$

$$\omega_- = \#\{\text{eig}(\Omega) < 0\}$$

$$\text{shift}(\lambda; A, H) = \#\{\text{spec}(A) < \lambda\} - \#\{\text{spec}(H) < \lambda\}$$

<u>Intuition</u>: $\lambda \nearrow$ eig $B(\lambda) \searrow$

have a 0 at spec(A)

have a pole at spec(H)

Matlab

<u>Proof</u>: Inertia $:= \{i_-, i_0, i_+\}$ $\quad \text{In}(T)$

$= \{\#\text{ neg eig}, \#\, 0 \text{ eig}, \#\text{ pos eig}\}$

Haynsworth formula:

$$\text{In}\begin{pmatrix} A & B \\ B^* & D \end{pmatrix} = \text{In}(D) + \text{In}\big(\underbrace{A - BD^{-1}B^*}_{\text{Schur Compl.}}\big)$$

if D-invertible

$$= \text{In}(A) + \text{In}(D - B^* A^{-1} B).$$

Consider

$$T = \begin{pmatrix} \Omega^{-1} & k \\ k^* & H-\lambda \end{pmatrix}$$

$$\ln(T) = \ln(\Omega^{-1}) + \ln\left(H - \lambda - k^*\Omega k\right)$$

$$= \ln \Omega + \ln(A-\lambda)$$

$$\ln(T) = \ln(H-\lambda) + \ln\left(\Omega^{-1} - k(H-\lambda)^{-1}k\right)$$

$$= \ln(H-\lambda) + \ln(B(\lambda))$$

$$\therefore \quad \ln(B(\lambda)) = \ln\Omega + \ln(A-\lambda) - \ln(H-\lambda) \quad \blacksquare$$

Lateral Variation Principle

Joint work with Peter Kuchment. For simplicity on matrices.

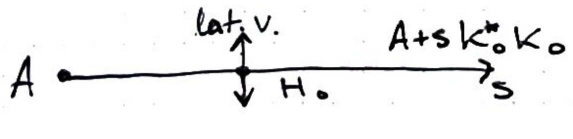
$H_0 : \mathcal{H} \to \mathcal{H}$ - Hermitian, $H_0 f^0 = \lambda^0 f^0$ λ^0 - simple

$K_0 : \mathcal{H} \to \mathcal{K}$ ($\mathcal{K}$ - aux. H. sp.) $K_0 f^0 = 0$

$$A = H_0 - K_0^* K_0$$

$$H(K) := A + K^*K \qquad (\text{in particular } H(K_0) = H_0)$$

For $K \approx K_0$ in $\mathcal{L}(\mathcal{H}, \mathcal{K})$ this is "lateral variation"



$\Lambda : \mathcal{L}(\mathcal{H}, \mathcal{K}) \to \mathbb{R}$ $\Lambda(K)$ - eigenvalue of $H(K)$ s.t.

$\Lambda(K_0) = \lambda^0$. Note: Λ smooth locally.

Thm: $K = K_0$ is a critical point of $\Lambda(K)$

The Morse index of this C.P. is equal to

$\mathrm{shift}(\lambda_0; A, H_0) = \#\{\mathrm{spec}(A) < \lambda^0\} - \#\{\mathrm{spec}(H_0) < \lambda^0\}$

(maybe $\mathcal{L}(\mathcal{H}, \mathcal{K})$ has too much dimension)

Thm: Let

$$H(\vec{s}) = A + \sum_{j=1}^{r} |u_j(s_j)\rangle\langle u_j(s_j)|$$

(here $u_j(0)$ is j-th row of K_0). Then $\vec{s} = 0$ is a C.P. of $\Lambda(\vec{s})$. Assume $\langle u_j'(0) | f^0 \rangle \neq 0 \ \forall j$.

Then Morse index of $\vec{s} = 0$ is $\mathrm{shift}(\lambda_0; A, H_0)$ and nullity $= \dim \mathrm{Ker}(A - \lambda^0) - 1$.

Proof outline: ① Pert. Formula

$$\lambda'' = \langle f^0, \left(H'' - 2H' P_\perp (H - \lambda^0)^{-1} P_\perp H'\right) f^0 \rangle$$

② $\rightsquigarrow$ $I - K_0 (H_0 - \lambda^0)^{-1} K_0^*$ Birman-Schw.!