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Analysis of Carleman Representation of Analytical Recursions

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We study a general method to map a nonlinear analytical recursion onto a linear one. The solution of the recursion is represented as a product of matrices whose elements depend only on the form of the recursion and not on initial conditions. First we consider the method for polynomial recursions of arbitrary degree and then the method is generalized to analytical recursions. Some properties of these matrices, such as the existence of an inverse matrix and diagonalization, are also studied.

1. INTRODUCTION

In our recent work [6, 7] we found a correspondence between the logistic map and an infinite-dimensional linear recursion. This correspondence was exploited to obtain a rather general method for the mapping of a polynomial recursion to a linear (but infinite-dimensional) one, which was reported in [7]. However, studying the literature, we found that our work is a rediscovery of a known approach called *Carleman linearization* [2, 5, 9, 10].

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Some questions of Carleman linearization, such as the applicability of the method for analytical recursions, are still open. In this paper we thoroughly study this topic, prove convergence of series used in the method, and reveal some useful properties of the linearization, which allows one to understand the correspondence between the original and linearized recursions.

The basis of the method is the construction of a *transfer matrix*, $\mathbf{T} = \{T_{ij}\}_{i,j=0}^{\infty}$. This allows one to represent the solution in the form

$$y_n = \tilde{\mathbf{e}}\mathbf{T}^n\mathbf{y},$$

where $\tilde{\mathbf{e}}$ is the row (transpose) vector $\{0, 1, 0, 0, \dots\}^T$, $\mathbf{y}$ is the column vector of initial values $\{1, y, y^2, y^3, \dots\}$, and $\mathbf{T}^n$ is the nth power of the infinite-dimensional matrix $\mathbf{T}$.

For introductory purposes we explain the method first for the polynomial recursions (see also [7]) and then prove its extension to the general case of analytical recursions.

2. POLYNOMIAL RECURSIONS

Here we consider a first-order recursion equation

$$y_{n+1} = P(y_n), \tag{1}$$

where $P(x)$ is a polynomial of degree m:

$$P(x) = \sum_{k=0}^{m} a_k x^k, \qquad a_m \neq 0. \tag{2}$$

Let $y_0 \equiv y$ be an initial value for the recursion (1). We denote by $\mathbf{y}$ the column vector of powers of y, $\mathbf{y} = \{y^j\}_{j=0}^{\infty}$, and set the vector $\tilde{\mathbf{e}}$ to be the row vector $\tilde{\mathbf{e}}^T = \{\delta_{j1}\}_{j=0}^{\infty}$. It should be emphasized that j runs from 0, since in the general case $a_0 \neq 0$. In this notation $\tilde{\mathbf{e}}\mathbf{y}$ is a scalar product that yields

$$\tilde{\mathbf{e}}\mathbf{y} = y. \tag{3}$$

THEOREM 1. *For any recursion of type Eq.* (1) *there exists a matrix* $\mathbf{T} = \{T_{jk}\}_{j,k=0}^{\infty}$, *defined by*

$$[P(y)]^j = \left(\sum_{i=0}^{m} a_i y^i\right)^j = \sum_{k=0}^{jm} T_{jk} y^k, \qquad j = 0, 1, \dots, \tag{4}$$

such that

$$y_n = \tilde{\mathbf{e}}\mathbf{T}^n\mathbf{y}. \tag{5}$$

The matrix power $\mathbf{T}^n$ *exists for all n and all the operations in the right-hand side of Eq.* (5) *are associative.*

Proof. For $n = 0$ the statement of the theorem is valid (see Eq. (3)). We introduce the column vector $\mathbf{y}_1 \stackrel{\text{def}}{=} \{y_1^j\}_{j=0}^{\infty}$, where $y_1 = P(y)$. Equation (4) implies that

$$\mathbf{y}_1 = \mathbf{T}\mathbf{y}. \tag{6}$$

Now, by analogy to Eq. (3), we have $y_1 = \tilde{\mathbf{e}}\mathbf{y}_1 = \tilde{\mathbf{e}}\mathbf{T}\mathbf{y}$. Therefore, the statement of the theorem is true for $n = 1$ as well.

Assume that Eq. (5) is valid for $n = l$ for any initial value y. Then y_{l+1} can be represented as $y_{l+1} = \tilde{\mathbf{e}}\mathbf{T}^l\mathbf{y}_1$, where $y_1 = P(y)$ is considered as a new initial value of the recursion. Then, using Eq. (6), one obtains

$$y_{l+1} = \tilde{\mathbf{e}}\mathbf{T}^l\mathbf{y}_1 = \tilde{\mathbf{e}}\mathbf{T}^l\mathbf{T}\mathbf{y} = \tilde{\mathbf{e}}\mathbf{T}^{l+1}\mathbf{y}.$$

Note that for j and k satisfying $k \geq jm$ we have $T_{jk} \equiv 0$. Therefore, each row is finite (i.e., there is only a finite number of nonzero matrix elements in each row). this proves the existence of powers of $\mathbf{T}$ and associativity in Eq. (5). Thus, the proof is complete.

EXAMPLE 1. As one can see, in the general case elements of the matrix $\mathbf{T}$ have a form of rather complicated sums. However, they reduce to a fairly simple expression, when the polynomial (2) has only two terms. To demonstrate this, let us consider the recursion (for a good reference on this subject, see [1])

$$y_{n+1} = \lambda y_n(1 - y_n) \qquad \text{with } y_0 \equiv y. \tag{7}$$

In this case one has

$$[P(y)]^j = (\lambda y - \lambda y^2)^j = \sum_{i=0}^{j} \lambda^{j-i}(-\lambda)^i \binom{j}{i} y^{(j-i)+2i}.$$

Denoting $i = k - j$, we have

$$[P(y)]^j = \lambda^j \sum_{k=j}^{2j} (-1)^{k-j} \binom{j}{k-j} y^k$$

and the matrix elements T_{jk} are

$$T_{jk} = (-1)^{k-j} \binom{j}{k-j} \lambda^j. \tag{8}$$

Thus, we recover a known result [6, 5] for the *logistic mapping*. Note that, as in Riordan [8], we use the definition of binomial coefficients such that $\binom{l}{m} = 0$ for the integers l and m, whenever $m < 0$ or $m > l$.

3. RECURSION SOLUTION FOR ARBITRARY ANALYTIC FUNCTIONS IN THE RHS

In this section we consider complex maps analytic in a disk $B_r = \{z: |z| < r\}$ centered at the origin. Real analytic maps are considered as a special case of complex maps.

Let the series

$$\sum_{k=0}^{\infty} a_k z^k \equiv f(z)$$

converge absolutely in the disk $B_r = \{z: |z| < r\}$. We construct the following series absolutely convergent in the disk B_r [4]:

$$\left[\sum_{k=0}^{\infty} a_k z^k\right]^j = a_0^j + \binom{j}{1} a_1 a_0^{j-1} z + \left[\binom{j}{1} a_2 j_0^{j-1} + \binom{j}{2} a_1^2 a_0^{j-2}\right] z^2 + \cdots$$
$$= \sum_{k=0}^{\infty} T_{jk} z^k.$$

DEFINITION 1. The matrix $\mathbf{T} = \{T_{jk}\}_{j,k=0}^{\infty}$ is said to be the transfer matrix of the analytic function $f(z)$ if $\sum_{k=0}^{\infty} T_{jk} z^k = [f(z)]^j$.

THEOREM 2. *Let $g(z)$ be analytic in the disk B_R, f: $B_r \to B_R$ be analytic in the disk B_r; let the matrices* $\mathbf{T}$ *and* $\mathbf{S}$ *be the transfer matrices of the functions $f(z)$ and $g(z)$. Then the matrix* $\mathbf{ST}$ *is the transfer matrix of the composition $g \circ f(z)$.*

Proof. First we prove the existence of the matrix product $\mathbf{ST}$. For arbitrary ρ, $0 < \rho < r$, one has $|f(z)| \leq R - \epsilon$ in the closed disk $|z| \leq \rho$, for some ϵ, $0 < \epsilon < R$. Thus, the power $[f(z)]^j$ satisfies $\|[f(z)]^j| \leq (R-\epsilon)^j$ and by Cauchy's inequality for the coefficients of the power series representation we have $|T_{jk}| \leq \rho^{-k}(R-\epsilon)^j$. Therefore, the series

$$|(\mathbf{ST})_{ik}| = \left|\sum_{j=0}^{\infty} S_{ij} T_{jk}\right| < \rho^{-k} \sum_{j=0}^{\infty} \left|S_{ij}(R-\epsilon)^j\right|$$

converges absolutely, since, by definition, the series $\sum_{j=0}^{\infty} S_{ij} z^j$ is absolutely convergent in B_R. Thus, the matrix product $\mathbf{ST}$ exists and, furthermore, we have

$$\sum_{k=0}^{\infty} \left|\sum_{j=0}^{\infty} S_{ij} T_{jk} z^k\right| \leq \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left|S_{ij} \frac{(R-\epsilon)^j}{\rho^k} z^k\right|$$
$$= \sum_{k=0}^{\infty} \frac{|z|^k}{\rho^k} \sum_{j=0}^{\infty} \left|S_{ij}(R-\epsilon)^j\right|$$

and the series $\sum_{k=0}^{\infty} (\mathbf{ST})_{ik} z^k$ converges absolutely in the disk B_ρ. Changing the order of summation [4], one has

$$\sum_{k=0}^{\infty} (ST)_{jk} z^k = \sum_{i=0}^{\infty} S_{ji} \sum_{k=0}^{\infty} T_{ik} z^k = \sum_{i=0}^{\infty} S_{ji} [f(z)]^i$$
$$= [g(f(z))]^j = [g \circ f]^j(z). \tag{9}$$

Considering the limit $\rho \to r$, we infer that Eq. (9) holds in the disk B_r. This observation completes the proof.

In this way we can write down the solution of the recursion

$$z_{n+1} = f(z_n) \qquad \text{with } z_0 \equiv z. \tag{10}$$

THEOREM 3. *Let f: $B_r \to B_r$ be analytic with transfer matrix* $\mathbf{T}$. *Then, for any initial point $z \in B_r$,*

$$z_n = \tilde{\mathbf{e}} \mathbf{T}^n \mathbf{z},$$

where, $\tilde{\mathbf{e}}^T = \{\delta_{i,1}\}_{i=0}^{\infty}$ *and* $\mathbf{z} = \{z^i\}_{i=0}^{\infty}$.

Since real analytic functions can be analytically continued to the complex plane, the above analysis can be employed in the real functions case as well. For practical use instead of the condition $g(z)$: $B_R \to B_r$ the stronger condition,

$$\sum_{i=0}^{\infty} |b_i| x^k < r \qquad \text{for } |x| < R,$$

can be used, where the b_i are coefficients of the series decomposition of the function f.

The following example presents a family of functions whose transfer matrices form is invariant under multiplication.

EXAMPLE 2. We consider the function $f(x) = ax/(b + x)$. The components of the transfer matrix $\mathbf{T}$ are

$$T_{jk} = \begin{cases} \delta_{j0}\delta_{k0} + \binom{k-1}{j-1} a^j b^{-k} & \text{for } 0 < j \le k, \\ 0 & \text{otherwise}. \end{cases} \tag{11}$$

If $g(x) = cx/(d + x)$ and **S** is the transfer matrix of $g(x)$, then the components of the matrix **ST** are

$$(ST)_{jk} = \sum_{i=0}^{\infty} S_{ji} T_{ik} = \delta_{j0}\,\delta_{k0} + \sum_{i=j}^{k} \binom{i-1}{j-1} c^j d^{-i} \binom{k-1}{i-1} a^i b^{-k}$$

$$= \delta_{j0}\,\delta_{k0} + \binom{k-1}{j-1} c^j b^{-1} \sum_{l=0}^{k-j} \left(\frac{a}{d}\right)^{l+j} \binom{k-j}{l}$$

$$= \delta_{j0}\,\delta_{k0} + \binom{k-1}{j-1} \left(\frac{ac}{a+b}\right)^{j} \left(\frac{bd}{a+b}\right)^{-k}.$$

One can see that the matrix elements $(ST)_{jk}$ have the same form as in Eq. (11).

4. OPERATIONS WITH TRANSFER MATRICES

An interesting question to ask now is: What is the inverse of the transfer matrix, for example, of matrix (8)? The answer to this question and a possible method to calculate a general form of elements of the inverse matrix gives

THEOREM 4. *Let* **T** *be the transfer matrix of the analytic function* $f(x)$, $f(0) = 0$, $f'(0) \neq 0$. *Then* $\mathbf{T}^{-1}$ *exists and is the transfer matrix of the inverse function* $f^{-1}(x)$.

Note that by the inverse function $f^{-1}(x)$ we understand the formal Taylor series which, under substitution, produces $f^{-1} \circ f(x) = f \circ f^{-1}(x) = x$. The series converges if and only if the function defined by the implicit function theorem as a solution of $f(x) = y$ for initial value $f(0) = 0$ is analytic at the point $y = 0$. Thus, only local information about the function $f(x)$ is used in the theorem.

Proof. The statement of the theorem implies that **T** is an upper triangular matrix. Indeed, $f(x)$ can be represented as $f(x) = x\phi(x)$ and therefore $[f(x)]^j = x^j[\phi(x)]^j$ and $T_{jk} = 0$ for $k < j$. Existence and uniqueness of the inverse matrix, which is also an upper triangular matrix, is proved in [3]. To prove the theorem, we consider a formal Taylor series

$$g(x) = \sum_{k=0}^{\infty} T_{1k}^{-1} x^k.$$

Now we can form a transfer matrix **S** of the function $g(x)$. Then $(\mathbf{ST})_{1j} = \delta_{1j}$, since the first row of the matrix **S** is the same as that of the matrix

$\mathbf{T}^{-1}$. On the other hand, Theorem 4 implies $g \circ f(x) = \sum_{k=0}^{\infty} (\mathbf{ST})_{1k} x^k = x$. Therefore, the powers $(g \circ f(x))^j$ are equal to $x^j = \sum_{k=0}^{\infty} \delta_{jk} x^k = \sum_{k=0}^{\infty} (\mathbf{ST})_{jk} x^k$, which implies $(\mathbf{ST})_{jk} = \delta_{jk}$. Thus, $\mathbf{ST}$ is the unity matrix, $\mathbf{S} = \mathbf{T}^{-1}$ and $g(x) = f^{-1}(x)$. The theorem is proved.

EXAMPLE 3. For the *logistic mapping* $f(x) = \lambda x(1 - x)$ we have $f^{-1}(y) = (1 + \sqrt{1 - 4y/\lambda})/2$ (we eliminate the second branch $(1 - \sqrt{1 - 4y/\lambda})/2$ of the inverse function because it violates the condition $f^{-1} \circ f(0) = 0$). To get an explicit form of elements T_{jk}^{-1}, one can notice that $f^{-1}(x)$ is simply related to the generating function $c(x) = (2x)^{-1}(1 - \sqrt{1 - 4x})$ of *Catalan numbers* [8]. Respectively, the powers $[c(x)]^j$ are decomposed with the aid of *ballot numbers* as follows:

$$[c(x)]^j = \sum_k a_{j+k-1,k} x^k,$$

where

$$a_{l-1,m} = \frac{l - m}{l + m} \binom{l + m}{m}$$

are ballot numbers. This relation immediately gives $T_{jk}^{-1} = \lambda^{-k} a_{k-1,k-j}$, i.e.,

$$T_{jk}^{-1} = \lambda^{-k} \frac{j}{2k - j} \binom{2k - j}{k - j}.$$

Writing down the orthogonality relation $\mathbf{T}^{-1}\mathbf{T} = \mathbf{E}$ in matrix components, one finds the combinatorial identity

$$\sum_k (-1)^{k-j} \frac{k}{2l - k} \binom{j}{k - j} \binom{2l - k}{l - k} = \delta_{j,l}$$

(see also [8]).

The additional condition $f'(0) \neq 1$ makes possible the diagonalization of a transfer matrix.

THEOREM 5. *Let* $\mathbf{T}$ *be the transfer matrix of the analytic function* $f(x)$, $f(0) = 0$, $f'(0) = \lambda \neq 0$, $\lambda \neq 1$. *Then transfer matrices* $\mathbf{D}$ *and* $\mathbf{D}^{-1}$ *exist such that*

$$\mathbf{T} = \mathbf{D}^{-1} \boldsymbol{\Lambda} \mathbf{D}, \tag{12}$$

where $\boldsymbol{\Lambda}$ *is a diagonal matrix with elements* $\Lambda_{jk} = \lambda^j \delta_{jk}$.

Proof. We rewrite Eq. (12) as follows:

$$\mathbf{DT} = \mathbf{\Lambda}\,\mathbf{D}$$

and note that the structure of matrices $\mathbf{T}$ and $\mathbf{\Lambda}$ is rather simple and the equation gives a straightforward algorithm for calculating the matrix $\mathbf{D}$. Indeed,

$$D_{jk} = \left(\lambda^j - T_{kk}\right)^{-1} \sum_{i=1}^{k-1} T_{ik} D_{ji}.$$

This relation defines off-diagonal elements of the matrix $\mathbf{D}$ while the diagonal ones remain arbitrary. Thus, we have a family of suitable upper triangular matrices. To prove that the family contains a transfer matrix, we fix the undetermined element D_{11} to be, say, $a \neq 0$ and, as before, consider a formal series

$$h(x) = \sum_{k=1}^{\infty} D_{1k} x^k.$$

This allows one to construct a transfer matrix $\mathbf{S}$ for the function $h(x)$ and, since $\mathbf{\Lambda}$ is also the transfer matrix of the function $\lambda(x) = \lambda x$, we can apply the reasoning from the previous proof to confirm that the equation

$$\mathbf{ST} = \mathbf{\Lambda}\,\mathbf{S}$$

holds. Indeed, the first rows of the matrices on the left- and right-hand sides are equal due to the fact that $S_{1k} = D_{1k}$. The rest of the equation is implied by Theorem 4. The theorem is proved.

COROLLARY 1. *Let f: $B_r \to B_r$ be an analytic function. Then for any initial point $z \in B_r$ we have the following representation*:

$$z_n = \tilde{\mathbf{e}}\mathbf{D}^{-1}\mathbf{\Lambda}^n\mathbf{Dz}.$$

EXAMPLE 4. The correspondences $\mathbf{T} \mapsto f(x)$, $\mathbf{D} \mapsto h(x)$, $\mathbf{D}^{-1} \mapsto h^{-1}(x)$, and $\mathbf{\Lambda} \mapsto \lambda(x)$ allow us to rewrite Eq. (12) in the "functional" form

$$f(x) = h^{-1}(\lambda h(x)),$$

where $h(x) = \sum_{i=1}^{\infty} D_{1k} x^k$. The elements D_{1k} obey the linear recursion

$$D_{1k} = (\lambda - \lambda^k)^{-1} \sum_{i=1}^{k-1} T_{ik} D_{1i}.$$

For some special cases this recursion admits an explicit solution for D_{1k} which gives us information about $h(x)$. For example, using this result, it is possible to recover the well-known representation of the logistic map for $\lambda = 4$:

$$4x(1-x) = \left(\sin 2\arcsin\sqrt{x}\right)^2 = \left(\sin\sqrt{4\left(\arcsin\sqrt{x}\right)^2}\right)^2,$$

which corresponds to $f(x) = 4x(1-x)$ and $h(x) = (\arcsin\sqrt{x})^2$ and implies the easy-to-analyze solution

$$f^n(x) = \left(\sin 2^n \arcsin\sqrt{x}\right)^2,$$

where $f^n(x)$ is nth iteration of the function $f(x)$.

5. SUMMARY

In this work we prove convergence of the series used in Carleman linearization of nonlinear analytic recursions. We establish a correspondence between analytic functions and infinite matrices of a certain type. This correspondence is proven to extend to operations with matrices. Namely, for an analytic function, satisfying certain conditions, and the corresponding matrix the following is true: the inverse matrix corresponds to the formal inverse series of the function. A similar result is obtained for the diagonalization of the matrix which corresponds to the formal representation

$$f(x) = h^{-1}(\lambda h(x))$$

of the given function $f(x)$, where $\lambda = f'(0)$ and the functions $h(x)$ and $h^{-1}(x)$ are given in the form of series. Convergence of this series is a possible subject for future research.

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REFERENCES

1. R. P. Agarwal, ‘‘Difference Equations and Inequalities,’’ Dekker, New York, 1992.
2. T. Carleman, Application de la théorie des équations intégrates linéaires aux systèmes d’équations differentielles nonlinéaires, *Acta Math.* **59** (1932), 63–68.
3. R. G. Cooke, ‘‘Infinite Matrices and Sequence Spaces,’’ Macmillan, London, 1950.

4. R. Courant, ‘‘Differential and Integral Calculus,’’ Vol. 1, Blackie & Son, London, 1963.
5. K. Kowalski and W.-H. Steeb, ‘‘Nonlinear Dynamical Systems and Carleman Linearization,’’ World Scientific, Singapore, 1991.
6. S. Rabinovich, G. Berkolaiko, S. Buldyrev, A. Shehter, and S. Havlin, Logistic map: An analytical solution, *Phys. A* **218** (1995), 457–460.
7. S. Rabinovich, G. Berkolaiko, and S. Havlin, Solving nonlinear recursions, *J. Math. Phys.* **37** (1996), 5828–5836.
8. J. Riordan, ‘‘Combinatorial Identities,’’ Wiley, London, 1968.
9. C. A. Tsiligiannis and G. Liberatos, Steady state bifurcations and exact multiplicity conditions via Carleman linearization, *J. Math. Anal. Appl.* **126** (1987), 143–160.
10. C. A. Tsiligiannis and G. Liberatos, Normal forms, resonance and bifurcation analysis via the Carleman linearization, *J. Math. Anal. Appl.* **139** (1989), 123–138.