Intermediate Wave Function Statistics

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We calculate statistical properties of the eigenfunctions of two quantum systems that exhibit intermediate spectral statistics: star graphs and Šeba billiards. First, we show that these eigenfunctions are not quantum ergodic, and calculate the corresponding limit distribution. Second, we find that they can be strongly scarred, in the case of star graphs by short (unstable) periodic orbits and, in the case of Šeba billiards, by certain families of orbits. We construct sequences of states which have such a limit. Our results are illustrated by numerical computations.

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It has been conjectured that the quantum spectral statistics of systems that are chaotic in the semiclassical limit are generically those of random matrix theory [1]. The behavior of the eigenfunctions of such systems is described by the semiclassical eigenfunction hypothesis [2,3], which implies that they equidistribute over the appropriate energy shell. This is in agreement with a theorem of Schnirelman [4], which implies equidistribution of almost all eigenstates on scales independent of $\hbar$, assuming only classical ergodicity. Such behavior is termed *quantum ergodicity*. This theorem still permits the possibility of a small number of states which do not equidistribute.

It has been suggested that some of these exceptional states may be "scarred" by short classical periodic orbits [5]. Further investigations [6–11] have distinguished between weak and strong scarring. Weak scarring relates to states averaged over energy windows that contain a semiclassically increasing number of levels, whereas strong scarring means that sequences of states can be constructed whose limit is wholly or in part supported by one or more periodic orbits. Thus far the only systems known rigorously to support strong scarring are the cat maps [12], which have nongeneric spectral statistics [13].

For systems that are classically integrable, it is expected that the quantum spectral statistics are Poissonian, i.e., those of independent random numbers [14]. The corresponding eigenfunctions semiclassically equidistribute on tori in phase space [15].

Recently, classes of systems which exhibit spectral statistics that are intermediate between random matrix and Poissonian have been discovered [16–18]. Two representative families of examples are Šeba billiards [19] and star graphs [20]. It was shown in [21] that these two systems have the same (intermediate) spectral statistics. We study the eigenfunction statistics of such systems. Specifically, given that these systems are not classically ergodic, we are interested in whether the eigenfunctions are quantum ergodic and whether they show strong scarring (that they exhibit weak scarring may be shown using the methods of [6]).

Star graphs are quantum graphs [22] that have one central vertex, and b outlying vertices each connected only to the central vertex [20]. For such graphs, the limit $b \to \infty$ is analogous to the semiclassical limit. To investigate the possibility of quantum ergodicity in this limit, we consider a graph with $b = \alpha v$ bonds, where $v \gg 1$, $\alpha > 1$, and introduce the observable B defined by

$$B = \begin{cases} 1 \text{ on bonds indexed } 1, \ldots, v, \\ 0 \text{ on bonds indexed } v+1, \ldots, b. \end{cases}$$

Thus B picks out a fraction α^{-1} of the bonds. Let ψ_n denote the wave function associated with the nth eigenstate. We calculate the probability distribution $P(R)$ for n chosen at random, that $\langle\psi_n|B|\psi_n\rangle$ is less than R, subject to some mild restrictions on the bond lengths. A system that exhibits quantum ergodicity would have

$$P(R) = \begin{cases} 0, & 0 \le R < \alpha^{-1}, \\ 1, & \alpha^{-1} \le R \le 1. \end{cases}$$

Our result [see Eq. (8) and Fig. 1] differs from this, proving that star graphs are not quantum ergodic. In

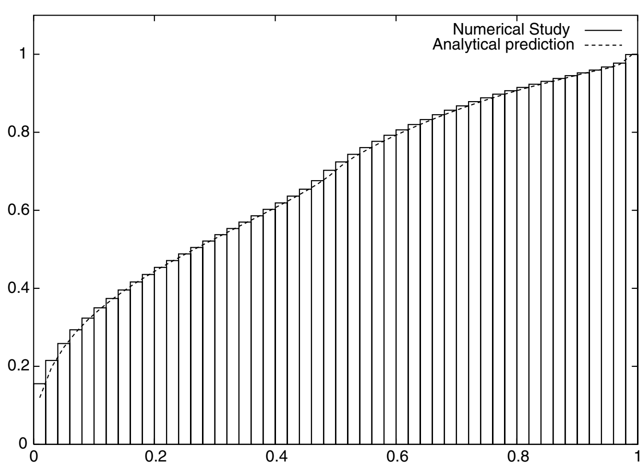

FIG. 1. Comparing $P(R)$, as given by (8), to a direct numerical computation for a star graph with 90 bonds when $\alpha = 3$.

fact, we are able to say more: For a fixed (finite) number of bonds, we explicitly find eigenstates that are strongly scarred along closed (unstable) orbits of the graph with period 2. This is the first class of examples showing generic (in this case intermediate) behavior in which strong scarring has been rigorously demonstrated.

The term *Šeba billiard* refers to any integrable quantum system that has been perturbed by the addition of a point singularity. We consider the specific example of a billiard on a torus. By exploiting the connection between Šeba billiards and star graphs [21] we argue that Šeba billiards are also not quantum ergodic and find states that appear to show behavior analogous to strong scarring, in this case by families of orbits.

We begin by describing how to calculate the probability distribution $P(R)$.

Eigenenergies of a star graph with b bonds are given by $E_n = k_n^2$, where k_n is the nth solution of $Z(k) = 0$ with

$$Z(k) = \sum_{j=1}^{b} \tan kL_j, \tag{1}$$

the individual bond lengths being denoted by $L_1, \ldots, L_b$. The component of the nth wave function on the ith bond of the graph is $\psi_{n,i}(x) = A_i(k_n)\cos k_n(x - L_i)$, where

$$A_i(k_n) = \frac{\sqrt{2}}{\cos k_n L_i (\sum_j L_j \sec^2 k_n L_j)^{1/2}}, \tag{2}$$

the sum being taken over all bonds. Then

$$\langle\psi_n|B|\psi_n\rangle = \frac{\sum_{i=1}^{v} L_i \sec^2 k_n L_i}{\sum_{j=1}^{b} L_j \sec^2 k_n L_j} + O(k_n^{-1}). \tag{3}$$

To calculate the distribution of values taken by this quantity we average over a large number of states, making the error term in (3) negligible. We choose incommensurate bond lengths from an interval $[\bar{L}, \bar{L} + \Delta L]$ that shrinks in such a way that $v\Delta L \to 0$ as $v \to \infty$. Thus we can replace L_i by $\bar{L}$ wherever it does not multiply k_n.

To evaluate a function $f(k)$ at the zeros of $Z(k)$ we integrate against the density of states, so

$$\frac{1}{N}\sum_{n=1}^{N} f(k_n) = \frac{1}{N}\int_0^{k_N} f(k) Z'(k)\delta(Z(k))dk,$$

where δ denotes the Dirac delta function. Writing the delta function in Fourier representation, $\delta(x) = (2\pi)^{-1}\int_{-\infty}^{\infty} e^{i\zeta x}d\zeta$, and taking the limit $N \to \infty$,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(k_n) = \frac{1}{2\pi\bar{d}}\lim_{K\to\infty}\frac{1}{K}\int_0^K\int_{-\infty}^{\infty} f(k)Z'(k)\exp[i\zeta Z(k)]d\zeta dk, \tag{4}$$

writing $K = k_N$ and using $k_N \approx N/\bar{d}$, where $\bar{d} = b\bar{L}/\pi$ is the mean density of states. We apply (4) with $f(k) = \exp[i\beta X_\eta(k)]$, where

$$X_\eta(k) = \frac{1}{v^2}\sum_{j=v+1}^{b}\sec^2 kL_j - \frac{\eta}{v^2}\sum_{i=1}^{v}\sec^2 kL_i \tag{5}$$

for β, η constants. This is related to the distribution of $\langle\psi_n|B|\psi_n\rangle$ by the fact that

$$\mathbb{P}(X_\eta(k_n) > 0) = \mathbb{P}(\langle\psi_n|B|\psi_n\rangle < R)$$

when R and η are related by $\eta = 1/R - 1$.

We observe that whenever k appears in (4) it is multiplied by a bond length, and is an argument of a π-periodic function. Since the bond lengths are incommensurate, the k integral can be rewritten as a multiple integral over the b variables $x_j = kL_j$. A similar argument was used in [23,24]. The integrand then factorizes, so that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(k_n) = \frac{1}{2\alpha v}\int_{-\infty}^{\infty} I_1 I_2^{v-1} I_3^{\alpha v - v} + (\alpha - 1) I_4 I_2^{v} I_3^{\alpha v - v - 1} d\zeta, \tag{6}$$

where

$$I_1 = \frac{1}{\pi}\int_0^{\pi}\sec^2 x \exp\left(\frac{i\zeta}{v}\tan x - \frac{i\beta\eta}{v^2}\sec^2 x\right)dx,$$

$$I_2 = \frac{1}{\pi}\int_0^{\pi}\exp\left(\frac{i\zeta}{v}\tan x - \frac{i\beta\eta}{v^2}\sec^2 x\right)dx,$$

I_3 is obtained by replacing β with $-\beta/\eta$ in I_2, and I_4 by making the same substitution in I_1. Techniques to analyze the asymptotics of these integrals were discussed in [24]. Using them we find that

$$I_1 \sim \frac{v}{\sqrt{\pi i\beta\eta}}\exp\left(\frac{i\zeta^2}{4\beta\eta}\right),$$

and

$$I_2^v \sim \exp\left[-\frac{2}{\sqrt{\pi}}\sqrt{i\beta\eta}\exp\left(\frac{i\zeta^2}{4\beta\eta}\right) - \zeta\,\mathrm{erf}\left(\frac{\zeta}{2\sqrt{i\beta\eta}}\right)\right],$$

as $v \to \infty$. Substituting the above into (6) and denoting the result $e(\beta)$, we arrive at

$$e(\beta) = \frac{1}{2\alpha}\int_{-\infty}^{\infty}\frac{1}{\sqrt{\beta}}T\left(\frac{\zeta}{\sqrt{\beta}}\right)\exp\left[-\sqrt{\beta}\,\tau\left(\frac{\zeta}{\sqrt{\beta}}\right)\right]d\zeta,$$

where

$$T(\xi) = \frac{1}{\sqrt{i\pi\eta}}\exp\left(\frac{i\xi^2}{4\eta}\right) + \frac{(\alpha - 1)}{\sqrt{-i\pi}}\exp\left(-\frac{i\xi^2}{4}\right)$$

and

$$\tau(\xi) = \frac{2}{\sqrt{\pi}}\sqrt{i\eta}\exp\left(\frac{i\xi^2}{4\eta}\right) + \xi\,\mathrm{erf}\left(\frac{\xi}{2\sqrt{i\eta}}\right) + \frac{2(\alpha - 1)}{\sqrt{i\pi}} \times \exp\left(-\frac{i\xi^2}{4}\right) + \xi(\alpha - 1)\mathrm{erf}\left(\frac{e^{i\pi/4}\xi}{2}\right).$$

The Fourier transform of $e(\beta)$ is the probability density function of $X_\eta(k_n)$ where the index of the state n is chosen at random. The probability distribution for $\langle\psi_n|B|\psi_n\rangle$ to be less than R is then given by

$$P(R) = \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^\infty e(\beta)e^{-i\beta\sigma}d\beta d\sigma \bigg|_{\eta=1/R-1}. \qquad (7)$$

The Fourier transform of $e(\beta)$ is

$$\frac{-1}{2\pi\alpha}\mathrm{Re}\int_{-\infty}^\infty T(\xi)\frac{\sqrt{\pi}\tau(\xi)}{2(i\sigma)^{3/2}}w\left(\frac{-\tau(\xi)}{2\sqrt{-i\sigma}}\right)d\xi,$$

having made the substitution $\xi = \zeta/\sqrt{\beta}$ and using the notation $w(z) = e^{-z^2}\mathrm{erfc}(-iz)$. Performing the σ integral in (7) gives, finally,

$$P(R) = \frac{1}{2} - \frac{1}{\pi\alpha}\mathrm{Im}\int_{-\infty}^\infty T(\xi)\log[\tau(\xi)]d\xi, \qquad (8)$$

with $\eta = 1/R - 1$, for $0 < R < 1$.

The results of numerical computations which support this calculation are shown in Figs. 1 and 2.

We now turn to constructing sequences of eigenstates on star graphs, when b is fixed, that are strongly scarred by certain short periodic orbits. (Note that on such graphs all orbits are unstable.) Our construction exploits the properties of the spectral determinant (1). The spectral determinant has poles at the points

$$P = \bigcup_{i=1}^b P_i \equiv \bigcup_{i=1}^b\left\{\frac{\pi/2 + \pi n}{L_i}: n \in \mathbb{Z}\right\}$$

(i.e., P is the union of the sets P_i). Since the derivative of $Z(k)$ is everywhere greater than zero, there is exactly one root of $Z(k) = 0$ between every two consecutive poles.

Given a small $\epsilon > 0$, which will control the quality of the scarred eigenstate, we can find a pole p_1 in the set P_1 satisfying the following properties: (a) there is a pole p_2 from P_2 within a distance ϵ of p_1 and (b) p_1 is approximately equidistant from the two nearest poles from P_i, for each $i > 2$. Because of the ergodic properties of the sequence P (assuming that the bond lengths L_i are incommensurate), the above situation occurs with nonzero frequency along the k axis.

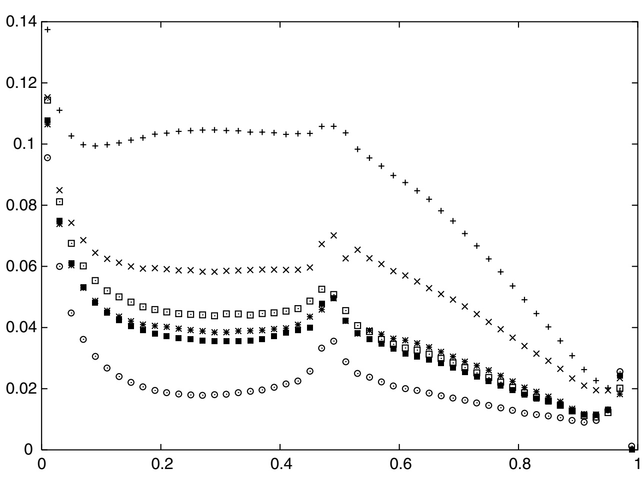

FIG. 2. Difference between $P(R)$ and numerics when $b = 15$ (+), 30 ($\times$), 45 (*), 60 (square with dot), 75 ($\blacksquare$), and 90 (circle with dot).

Denote the root squeezed between p_1 and p_2 by k'. Then $\cos k'L_i$ is of the order of ϵ when $i = 1, 2$ and is of the order of 1 otherwise. Going back to the eigenstate formula (2), we see that

$$\frac{A_{1,2}(k')}{A_i(k')} = O(\epsilon^{-1}) \qquad \text{for } i > 2;$$

that is, the amplitude of the k' eigenstate on the bonds 1 and 2 is ϵ^{-1} times stronger than on any other bond. By selecting suitably small ϵ one can find eigenstates localized on any two given bonds to any precision. Understandably, higher precision leads to a smaller frequency of the scarred eigenstates. In fact, the frequency is proportional to ϵ.

Since $Z(k') = 0$ it follows that $A_1(k') \approx A_2(k')$ which provides an explanation for the visible singularity at $R = 1/2$ in the difference between $P(R)$ for finite b and its limiting form (see Fig. 2). This singularity corresponds to the eigenstates localized on bonds e and e' such that e is picked out by the observable B and e' is not.

The above construction can be generalized to produce eigenstates localized on any number $j \geq 2$ of bonds. However, once $j > 2$, the amplitudes on the j bonds are generally not equal, which explains the lack of singularities at rational fractions other than 1/2. Finally, the singularities at $R = 0$ and 1 correspond to the cases when the eigenstates are localized fully outside ($R = 0$) or inside ($R = 1$) the v bonds picked out by B.

The preceding calculations can be made rigorous. We defer the details to [25].

In [24] it was suggested that the squares of the coefficients, c_i^2, of the eigenfunctions of Šeba billiards expressed in the basis of states of the unperturbed billiard

$$|\psi\rangle = \sum_i c_i|\psi_i^{(0)}\rangle \qquad (9)$$

are distributed in the same way as the square of the maximum norm on a single bond of a star graph in the limit as $v \to \infty$. This conjecture was supported by numerical evidence. We extend this analogy to interpret the above results in terms of the Šeba billiard. Since the quantity in (3) is similar to a sum of norms of eigenfunctions on a fraction of bonds, we conjecture that the sum of the squares of a fraction α^{-1} of the coefficients has probability distribution $P(R)$. To elucidate this idea, consider preparing a Šeba-type system in a randomly chosen eigenstate. The perturbation is then removed instantaneously, and a measurement of the energy is made. What is the distribution (with respect to the choice of initial state) of the probability that the measured energy is one of a given fraction α^{-1} of the energy levels of the unperturbed system? The answer is the distribution function in (8). If

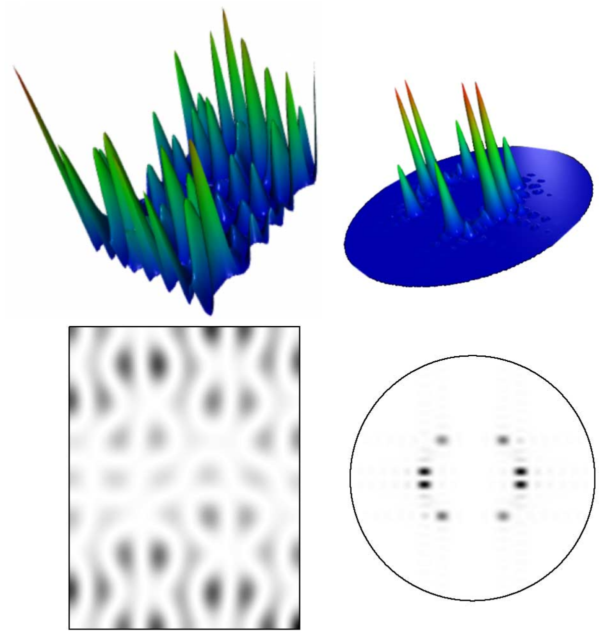

FIG. 3 (color online). The wave density of the 55th eigenfunction of the Šeba billiard, in position (left) and momentum (right) space. Intensity plots are shown below the three-dimensional plots; greater probability density is encoded as darker points. In this example $\gamma = (\sqrt{5}+1)/2$.

the eigenfunctions of the billiard were asymptotically equidistributed this probability distribution would be a unit step function at $R = 1/\alpha$.

Energy levels of a Šeba billiard interlace with energy levels of the original unperturbed system in much the same way that momenta of star graphs interlace with poles of the function $Z(k)$. We consider a Neumann billiard in a rectangle with aspect ratio $\gamma^{1/2}$, perturbed by a point singularity at the origin. Eigenstates of this system can be expanded as

$$|\psi_n(\mathbf{x})\rangle = A_n \sum_{i,j} \frac{|\psi_{i,j}^{(0)}(\mathbf{x})\rangle}{E_{i,j}^{(0)} - E_n}, \tag{10}$$

where A_n is a normalization constant, the energy levels of the Neumann billiard are $E_{i,j}^{(0)} = 4\pi^2\gamma^{-1/2}(i^2 + \gamma j^2)$, and $|\psi_{i,j}^{(0)}\rangle$ are the corresponding eigenfunctions. It is well known that these unperturbed eigenfunctions are localized in momentum space. We therefore expect to find states of the Šeba billiard that exhibit structures analogous to scars in momentum space when their energy is between two closely spaced levels of the unperturbed billiard. In fact such states will scar in two directions in momentum space, corresponding to the two unperturbed eigenstates closest in energy to the state in question. These scars are supported by families of orbits corresponding to tori in the unperturbed system. Note however that torus quantization itself does not apply. It is in this sense that the structures are analogous to scars.

Figure 3 shows the 55th state of the Šeba billiard described above, with the scatterer placed at the origin. Although there is no clear localization evident in position representation, the momentum representation clearly shows localization in two directions.

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