

Lecture 10

Overview

Chapter 9 Chapter 9 is about polynomial rings, one of the two culminating concepts of the course. All of us have done high school algebra, and chapter 9 contains the underpinnings of what we have done computationally for years or decades. We will learn about general polynomial rings and then apply that knowledge to prove that what we teach in Algebra I and II in high school is indeed true. I've included a reading I wrote for undergraduates that gives more about factoring over the integers, rationals, reals, and complexes than is contained in most undergraduate texts. There are some theorems that don't appear in standard texts as well as some proofs that are hard to locate. For those teaching high school, the extra handout is vital to not only your understanding of what you're teaching but also to the computational aspects. I did not go into synthetic division but, if you don't know how to do it already, I suggest you look it up on the web as it is a great time saver.

Chapter 13 The big culmination of the course is geometric in question and algebraic in answer. The ancient Greeks were big on constructions using straight-edge and compass. They had three unanswered questions that frustrated them no end. The first is how to build a cube with double the volume of a given cube. Another is how to trisect any angle. The third is to construct a square with the area of a given circle. In chapter 13 we will learn exactly which real numbers can be constructed with a straight-edge and compass, show that they form a subfield of the real numbers, and answer the three big questions.

9.1 This section contains definitions you've been using for years. Only proposition 2 is new but is logical.

9.1.1: Let $p(x, y, z) = 2x^2y - 3xy^3z + 4y^2z^5$ and $q(x, y, z) = 7x^2 + 5x^2y^3z^4 - 3x^2z^3$ be polynomials in $\mathbb{Z}[x, y, z]$.

9.1.1a: $p(x, y, z) = (2y)x^2 - (3y^3z)x + 4y^2z^5$, $q(x, y, z) = (7 + 5y^3z^4 - 3z^3)x^2$.

9.1.1b: The degree of p is 7 and the degree of q is 9.

9.1.1c: The x degree of p is 2 and of q is 2. The y degree of p is 3 as if the y degree of q . The z degree of p is 5 and of 1 is 4.

9.1.1d: $p(x)q(x) = (2x^2y - 3xy^3z + 4y^2z^5)(7x^2 + 5x^2y^3z^4 - 3x^2z^3) = 14x^4y + 10x^4y^4z^4 - 6x^4yz^3 - 21x^3y^3z - 15x^3y^6z^5 + 9x^3y^3z^4 + 28x^2y^2z^5 + 20x^2y^5z^9 - 12x^2y^2z^8$. Thus the x degree of pq is 4, the y degree 6, and the z degree 9.

9.1.1e: $pq(x, y, z) = 20x^2y^5z^9 - 12x^2y^2z^8 + (28x^2y^2 - 15x^3y^6)z^5 + (10x^4y^4 + 9x^3y^3)z^4 - 6x^4yz^3 - 21x^3y^3z + 14x^4y$.

9.1.3: Suppose R is a commutative ring and that x_1, c_2, \dots, x_n are independent variables over R . Let $\pi \in S_n$. Define $\rho : R[x_1, \dots, x_n] \rightarrow R[x_{\pi(1)}, \dots, x_{\pi(n)}]$ by $\rho(f(x_1, \dots, x_n)) = f(x_{\pi(1)}, \dots, x_{\pi(n)})$. Note that ρ is surjective by definition. Then $\rho(f(x_1, \dots, x_n) + g(x_1, \dots, x_n)) = \rho((f + g)(x_1, \dots, x_n)) = (f + g)(x_{\pi(1)}, \dots, x_{\pi(n)}) = f(x_{\pi(1)}, \dots, x_{\pi(n)}) + g(x_{\pi(1)}, \dots, x_{\pi(n)}) = \rho(f(x_1, \dots, x_n)) + \rho(g(x_1, \dots, x_n))$. Similarly, $\rho(f(x_1, \dots, x_n)g(x_1, \dots, x_n)) = \rho((fg)(x_1, \dots, x_n)) = (fg)(x_{\pi(1)}, \dots, x_{\pi(n)}) = f(x_{\pi(1)}, \dots, x_{\pi(n)})g(x_{\pi(1)}, \dots, x_{\pi(n)}) = \rho(f(x_1, \dots, x_n))\rho(g(x_1, \dots, x_n))$. Therefore, ρ is a surjective homomorphism. Suppose $0 = \rho(f(x_1, \dots, x_n)) = f(x_{\pi(1)}, \dots, x_{\pi(n)})$. Then $f = 0$, whence ρ is an isomorphism.

9.1.6: Suppose $(x, y) \in \mathbb{Q}[x, y]$ is principal. Then $(x, y) = (f(x, y))$. Thus $x = f(x, y)g(x, y)$ and $y = f(x, y)h(x, y)$. Since the degree of x and y is 1, $\deg(f) + \deg(g) = 1 = \deg(f) + \deg(h)$. If $\deg(f) = 0$, then $f(x, y) = c \in \mathbb{Q}$ which is a unit, so $(f) = \mathbb{Q}[x, y]$. Thus $\deg(f) = 1$, so $f(x, y) = ax + by$, $a, b \in \mathbb{Q}$. But then $\deg(g) = 0$, so x and y are associates of f , whence each other. But $x \neq ay \in \mathbb{Q}[x]$. Contradiction. Therefore (x, y) is not a principal ideal.

9.2 The only theorem of this section is one we all know, namely, the Euclidean Algorithm for $F[x]$ where F is a field. So mostly we need to work some problems here and at home.

9.2.3: Let $f(x) \in F[x]$. The $F[x]/(f(x))$ is a field if and only if $(f(x))$ is a maximal ideal if and only if $(f(x))$ is a prime ideal since $F[x]$ is a PID if and only if $f(x)$ is irreducible (8.3 Proposition 11).

9.2.5: Suppose F is a field and $p(x) \in F[x]$. Then $p(x) = af_1^{m_1} \cdots f_n^{m_n}$ where f_i are distinct, monic, irreducible polynomials and $a \in F$. Let $\pi : F[x] \rightarrow F[x]/(p(x))$ be the standard projection homomorphism. If \bar{I} is an ideal in $F[x]/(p(x))$, then $\pi^{-1}(\bar{I}) = I = (g(x))$ and $(p(x)) \subseteq (g(x))$. Therefore, $g(x)|p(x)$ and $\bar{I} = \overline{(g(x))}$. Thus all ideals of $F[x]/(p(x))$ are principal ideals generated by factors of $p(x)$.