Diffusion Coefficients Estimation for Elliptic Equations

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Abstract

This paper considers the Dirichlet problem

$$-\mathrm{div}(a\nabla u_a) = f \quad \text{on} \quad D, \qquad u_a = 0 \quad \text{on } \partial D,$$

for a Lipschitz domain $D \subset \mathbb{R}^d$, where a is a scalar diffusion function. For a fixed f, we discuss under which conditions is a uniquely determined and when can a be stably recovered from the knowledge of u_a. A first result is that whenever $a \in H^1(D)$, with $0 < \lambda \le a \le \Lambda$ on D, and $f \in L_\infty(D)$ is strictly positive, then

$$\|a - b\|_{L_2(D)} \le C\|u_a - u_b\|_{H_0^1(D)}^{1/6}.$$

More generally, it is shown that the assumption $a \in H^1(D)$ can be weakened to $a \in H^s(D)$, for certain $s < 1$, at the expense of lowering the exponent $1/6$ to a value that depends on s.

1 Introduction

Let D be a bounded domain (open, connected set) in $\mathbb{R}^d$, $d \ge 2$. We assume throughout the paper that, at a minimum, D is Lipschitz. We define the set of scalar diffusion coefficients

$$\mathcal{A} := \{a \in L_\infty(D) \ : \ \lambda \le a \le \Lambda\}, \tag{1.1}$$

where λ, Λ are fixed positive constants. For $f \in H^{-1}(D)$ (the dual of $H_0^1(D)$) and $a \in \mathcal{A}$, we consider the elliptic problem

$$-\mathrm{div}(a\nabla u_a) = f \quad \text{on} \quad D, \qquad u_a = 0 \quad \text{on } \partial D, \tag{1.2}$$

written in the usual weak form: $u_a \in H_0^1(D)$ is such that

$$\int_D a\nabla u_a \cdot \nabla v = \langle f, v\rangle_{H^{-1}(D),H_0^1(D)}, \qquad v \in H_0^1(D). \tag{1.3}$$

Here $H_0^1(D)$ is equipped with the norm $\|v\|_{H_0^1(D)} = \|\nabla v\|_{L_2(D)}$. The Lax-Milgram theory guarantees that there is a unique solution $u_a \in H_0^1(D)$ of the above problem.

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The main interest of the present paper is to understand, for a given f, the conditions under which the diffusion coefficient a is uniquely determined from the solution u_a to (1.3), and if so, whether a can be stably recovered if u_a is known. After having fixed f, we systematically denote by u_a the solution of (1.3). We are therefore interested in the stable inversion of the map

$$a \mapsto u_a \tag{1.4}$$

which acts from $\mathcal{A}$ to $H_0^1(D)$. By stability, we mean that when u_b is close to u_a, say in the $H_0^1(D)$ norm, then it follows that b is close to a in some appropriate $L_p(D)$ norm. The results of this paper will prove such stable inversion but only when certain restrictions are placed on the right side f and further only when the map (1.4) is restricted to certain subclasses of $\mathcal{A}$.

Problems of this type are referred to as parameter estimation, or the identifiability problem in the inverse problems literature, see e.g. [6, 1, 19, 16, 15] and the references therein. Parameter estimation/identification for elliptic partial differential equations and their numerical recovery from the (partial) knowledge of u_a is an extensively studied subject that has been formulated in several settings. Examples of such settings are the identifiability of the diffusion coefficient a in the problem $-\mathrm{div}(a\nabla u) = 0$ from the Neumann boundary data g on ∂D, see [17], or the recovery of a from the solution u to equation (1.2) supplemented by Dirichlet boundary data, see [15].

Let us make a few elementary remarks about the Dirichlet boundary data setting studied here. These remarks extend to other settings as well. For $a \in \mathcal{A}$, we denote by T_a the elliptic operator $u \mapsto -\mathrm{div}(a\nabla u)$ which is an isomorphism from $H_0^1(D)$ to $H^{-1}(D)$, and by S_a its inverse. Then, it is not difficult to check, see Lemma 2.1 in §2, that the map $a \mapsto S_a$ is bi-Lipschitz from $L_\infty(D)$ to $\mathcal{L}(H^{-1}(D), H_0^1(D))$, with bounds

$$\lambda^2 \|S_a - S_b\|_{\mathcal{L}(H^{-1}(D),H_0^1(D))} \le \|a-b\|_{L_\infty(D)} \le \Lambda^2 \|S_a - S_b\|_{\mathcal{L}(H^{-1}(D),H_0^1(D))}, \quad a, b \in \mathcal{A}. \tag{1.5}$$

Therefore, any $a \in \mathcal{A}$ can be stably identified in the L_∞ norm from the inverse operator S_a, that is, if we knew the solution to (1.3) for *all* possible right sides then a is uniquely determined. Note that (1.5) also means that, for any $a, b \in \mathcal{A}$, there exists a right side $f = f(a, b)$, with $\|f\|_{H^{-1}(D)} = 1$, for which we have the Lipschitz bound

$$\|a - b\|_{L_\infty(D)} \le \Lambda^2 \|u_a - u_b\|_{H_0^1(D)}. \tag{1.6}$$

The f for which (1.6) holds depends on a and b. Our objective is to fix one right side f and study the stable identifiability of a from u_a. It is well known that identifiabiliy cannot hold for an arbitrary right side f, even when f is smooth. For example, if u is any function in $H_0^1(D)$ such that ∇u is identically 0 on an open set $D_0 \subset D$, then setting $f = -\mathrm{div}(a\nabla u)$ for some fixed $a \in \mathcal{A}$, we find that $u = u_a = u_b$ for any $b \in \mathcal{A}$ which agrees with a on $D \setminus D_0$. The above example can be avoided by assuming that f is strictly positive. However, even in the case that f is strictly positive, we do not know a proof of identifiabilty under the general assumption that $a \in \mathcal{A}$, except in the univariate setting.

In this paper, we show that for strictly positive $f \in L_\infty(D)$, identifiability and stability hold, for a certain range of $s > 0$, in the restricted classes $\mathcal{A}_s \subset \mathcal{A}$, where

$$\mathcal{A}_s := \mathcal{A}_{s,M} := \{a \in \mathcal{A} \ : \ \|a\|_{H^s(D)} \le M\}. \tag{1.7}$$

Here, $M > 0$ is arbitrary but enters in the value of the stability constants. Under such conditions, we establish results of the form (see for example 4.6)

$$\|a - b\|_{L_2(D)} \le C \|u_a - u_b\|^{\alpha}_{H_0^1(D)}, \quad a, b \in \mathcal{A}_s, \tag{1.8}$$

where the exponent $0 < \alpha < 1$ depends on s and the constant C depends on $\lambda, \Lambda, \alpha, M, D, f$. Some elementary observations in the univariate case, see §6, show that when $f = 1$ and $\mathcal{A}_s$ includes discontinuous functions, the exponent α cannot be larger than $1/3$.

There are several existing approaches to establish identifiability. For the most part, they are developed for the Neumann problem

$$-\mathrm{div}(a\nabla u_a) = f \quad \text{on } D, \quad a\frac{\partial u_a}{\partial n} = g \quad \text{on } \partial D, \tag{1.9}$$

where n denotes the outward pointing normal to ∂D. Some approaches use singular perturbation arguments, see [2], or the long time behavior of the corresponding unsteady equations, see [14]. Some results rely on the observation that once $u = u_a$ is given, (1.9) may be viewed as a transport equation for the diffusion a, see [22, 23], and the identifiability of a from u_a is proven under the assumptions that a is prescribed on the inflow boundary (the portion of the boundary where $\frac{\partial u_a}{\partial n} < 0$) and

$$\inf_D \max\{|\nabla u_a|, \Delta u_a\} > 0. \tag{1.10}$$

Other approaches to identifiability use variational methods, see [16], or least-squares techniques, see [11, 18, 20, 9]. These approaches impose strong regularity assumptions on a and u_a as well as the assumption

$$\nabla u_a \cdot \tau > 0, \tag{1.11}$$

for a given $\tau \in \mathbb{R}^d$, or the less restrictive condition (1.10). Rather than directly proving a stability estimate, they derive numerical methods for actually finding the diffusion coefficient a from the solution u_a over triangulation $\mathcal{T}_h$ of D with mesh size h. One typical reconstruction estimate, see Theorem 1 in [9], is the following. Let $r \geq 1$ and let A_h and V_h be the sets of continuous piecewise polynomials on $\mathcal{T}_h$ of degree r and $r+1$, respectively. If (1.11) holds, and if $u_a \in W^{r+3}(L_\infty(D))$ and $a \in H^{r+1}(D)$, then

$$\|a - a_h\|_{L_2(D)} \leq C\left(h^r + \|u_a - u^{ob}\|_{L_2(D)} h^{-2}\right), \tag{1.12}$$

where $u^{ob} \in L_2(D)$ is an observation of u_a, and $a_h \in A_h$ is a numerical reconstruction of a via least squares type approach from the observation u^{ob}. As shown in Remark 4.4, the inequality (1.12) leads to a stability estimate of the form

$$\|a - b\|_{L_2(D)} \leq C\|u_a - u_b\|^{\alpha}_{L_2(D)}, \quad \alpha := \frac{r}{r+2}, \quad a, b \in \mathcal{A}_{r+1}, \tag{1.13}$$

whenever in addition $u_a, u_b \in W^{r+3}(L_\infty(D))$ and condition (1.11) holds. Note that α approaches 1 as $r \to \infty$.

In summary, the majority of the existing stability estimates are derived for solutions to the Neumann problem (1.9). As illustrated by (1.13), they rely on strong regularity assumptions on the diffusion coefficients a and on the solutions u_a, as well as conditions on u_a such as (1.11) or (1.10). However, one should note that high order smoothness of u_a generally does not hold, even for smooth a and f, when the domain D does not have a smooth boundary.

In this paper, we pursue a variational approach, where we use appropriate test functions v in (1.3) to derive continuous dependence estimates. We combine these with known elliptic regularity results and obtain direct comparison between $\|a - b\|_{L_2(D)}$ and $\|\nabla u_a - \nabla u_b\|_{L_2(D)}$ under milder smoothness assumptions for the diffusion coefficient a, the domain D, and on the right side f, and with *no additional smoothness assumptions* on u_a and no conditions such as (1.10) or (1.11).

We mention two special cases of our results. The first, see Corollary 3.8, says that if D is an arbitrary Lipschitz domain, then for any $f \in L_\infty(D)$ satisfying $f \geq c_f > 0$ on D, we have the stability bound

$$\|a-b\|_{L_2(D)} \leq C\|u_a-u_b\|_{H_0^1(D)}^{1/6}, \quad a,b \in \mathcal{A}_1. \tag{1.14}$$

We can weaken the smoothness assumption to the classes $\mathcal{A}_s$, for $s<1$. We have two types of results. In Corollary 4.5, we prove estimates of the form

$$\|a-b\|_{L_2(D)} \leq C\|u_a-u_b\|_{H_0^1(D)}^{\alpha}, \quad a,b \in \mathcal{A}_s, \tag{1.15}$$

with α depending on s, for all $1/2 < s < 1$ under the additional assumption that the diffusion coefficients are in VMO and the domain D is C^1. In Corollary 4.6, we prove for a general Lipschitz domain D, that (1.15) holds for a certain range of $s^* < s < 1$ where we do not require the diffusion coefficients are in VMO but now s^* depends on properties of the domain D.

Estimates like (1.13) have a weaker norm on the right side then those in our results. However, let us remark that any such estimate can be transformed into an estimate between $\|a-b\|_{L_2(D)}$ and $\|u_a-u_b\|_{L_2(D)}$, if the solutions u_a and u_b have more regularity such as the condition u_a and u_b belong to $H^{1+t}(D)$ for some $t>0$. For this, one uses the interpolation inequality

$$\|v\|_{H^1(D)} \leq C\|v\|_{L_2(D)}^{\theta}\|v\|_{H^{1+t}(D)}^{1-\theta}, \quad v \in H^{1+t}(D), \tag{1.16}$$

where $\theta := \frac{t}{1+t}$. Hence, assuming that $u_a, u_b \in H^{1+t}(D)$, one has

$$\|u_a-u_b\|_{H_0^1(D)} \leq C\|u_a-u_b\|_{L_2(D)}^{\theta}, \tag{1.17}$$

which combined with (1.15) leads to

$$\|a-b\|_{L_2(D)} \leq C\|u_a-u_b\|_{L_2(D)}^{\alpha\theta}. \tag{1.18}$$

Let us additionally note that as $r \to \infty$, the result in (1.13) leads to better exponents then in our results. This is caused, at least in part, by the fact that our starting point is (1.14) which does not use higher smoothness than $a, b \in H^1(D)$.

Our paper is organized as follows. In §2, we use a variational approach to establish a weighted L_2 estimate

$$\|a-b\|_{L_2(w,D)} \leq C\|u_a-u_b\|_{H_0^1(D)}^{1/2}, \quad a,b \in \mathcal{A}_1, \tag{1.19}$$

where the weight is given by $w = a|\nabla u_a|^2 + f u_a$. In order to remove the weight in the above estimate, in §3, we introduce the *positivity condition*

$$\mathbf{PC}(\beta): \qquad a|\nabla u_a(x)|^2 + f(x)u_a(x) \geq c\,\mathrm{dist}(x,\partial D)^{\beta}, \quad \text{a.e on } D, \tag{1.20}$$

for some $\beta \geq 0$ and $c>0$, see Definition 3.1. Under this condition, we prove the stability estimate

$$\|a-b\|_{L_2(D)} \leq C\|u_a-u_b\|_{H_0^1(D)}^{\alpha}, \quad \alpha = \frac{1}{2(\beta+1)}, \quad a,b \in \mathcal{A}_1. \tag{1.21}$$

Notice that the smaller the β, the stronger the stability estimate.

We go further in §3 and investigate which regularity assumptions guarantee that the positivity condition $\mathbf{PC}(\beta)$ holds, and thereby obtain results in which this condition is not assumed but rather implied by the regularity assumptions on a. In particular, we prove that condition $\mathbf{PC}(2)$ is valid for the entire class $a \in \mathcal{A}$, provided $f \in L_2(D)$ with $f \geq c_f > 0$. We also show that certain smoothness conditions on the diffusion coefficient a, the right side f, and the domain D imply the positivity condition $\mathbf{PC}(0)$. However, as discussed in §3.1.2, $\mathbf{PC}(\beta)$ does not generally hold for $\beta < 2$ without additional regularity assumptions on the domain D.

In §4, we use interpolation arguments to obtain results under weaker assumptions than $a, b \in \mathcal{A}_1$. In §5, we provide stability estimates in the case when a is piecewise constant which is not covered by our general stability results. Finally, in §6, we provide stability estimates in the one dimensional case for $f = 1$ and general $a, b \in \mathcal{A}$. In this simple case, we also establish converse estimates which show that the Hölder exponent α in (1.8) cannot be above the value $\frac{1}{3}$ when a and b have low smoothness.

We conclude this introduction by stating some natural open problems in relation with this paper:

(i) While the identifiability problem is solved in this paper under mild regularity assumptions, it is still not known whether there exists an f for which the mapping $a \mapsto u_a$ is injective from $\mathcal{A}$ to $H_0^1(D)$ for a general multivariate Lipschitz domain D.

(ii) The best possible value $\alpha^* = \alpha^*(s)$ of the exponent α in (1.8) is generally unknown. In particular, we do not know if there exists some finite s_0 such that $\alpha^*(s) = 1$ when $s \geq s_0$.

(iii) All our results are confined to the case of scalar diffusion coefficients. Similar stability estimates for matricial coefficients would require considering the solutions u_a and u_b for more than one right side f. However we are not aware of results that solve this question.

2 First estimates

We begin by briefly discussing the stability properties of the maps $a \mapsto T_a$ and $a \mapsto S_a$.

Lemma 2.1. *For any $a, b \in \mathcal{A}$, we have*

$$\|T_a - T_b\|_{\mathcal{L}(H_0^1(D), H^{-1}(D))} = \|a - b\|_{L_\infty(D)}, \tag{2.1}$$

and

$$\lambda^2 \|S_a - S_b\|_{\mathcal{L}(H^{-1}(D), H_0^1(D))} \leq \|a - b\|_{L_\infty(D)} \leq \Lambda^2 \|S_a - S_b\|_{\mathcal{L}(H^{-1}(D), H_0^1(D))}. \tag{2.2}$$

Proof: For the proof of (2.1), we observe on the one hand that

$$|\langle (T_a - T_b)u, v\rangle_{H^{-1}(D), H_0^1(D)}| \leq \|a - b\|_{L_\infty(D)} \|u\|_{H_0^1(D)} \|v\|_{H_0^1(D)}, \quad u, v \in H_0^1(D), \tag{2.3}$$

which shows that the right quantity dominates the left one in (2.1). On the other hand, for any $x \in D$ and $\varepsilon > 0$ small enough so that the open ball $B(x, \varepsilon)$ of radius ε centered at x is a subset of D, we consider the function $u = u_{x,\varepsilon}$ defined by

$$u(y) = \max\{0, 1 - \varepsilon^{-1}|x - y|\}. \tag{2.4}$$

For such a function, we find that

$$\langle (T_a - T_b)u, u\rangle_{H^{-1}(D),H_0^1(D)} = C_{x,\varepsilon}\|u\|^2_{H_0^1(D)}, \quad C_{x,\varepsilon} := |B(x,\varepsilon)|^{-1} \int_{B(x,\varepsilon)} (a(y) - b(y))dy. \tag{2.5}$$

By Lebesgue theorem, this shows that

$$\|T_a - T_b\|_{\mathcal{L}(H_0^1(D),H^{-1}(D))} \geq a(x) - b(x), \quad \text{a.e. } x \in D. \tag{2.6}$$

Since we can interchange the role of a and b, this shows that the left quantity dominates the right one in (2.1). For the proof of (2.2), we observe that $T_a(S_a - S_b)T_b = T_b - T_a$, which yields

$$\lambda^2\|S_a - S_b\|_{\mathcal{L}(H^{-1}(D),H_0^1(D))} \leq \|T_a - T_b\|_{\mathcal{L}(H_0^1(D),H^{-1}(D))} \leq \Lambda^2\|S_a - S_b\|_{\mathcal{L}(H^{-1}(D),H_0^1(D))}, \quad a, b \in \mathcal{A}. \tag{2.7}$$

Combined with (2.1), this gives (2.2). □

As observed in the introduction, the above result does not meet our objective, since we want to fix the right side $f \in H^{-1}(D)$ and then study the stable identifiability of a from u_a for all $a \in \mathcal{A}$. For such an f, let u_a, u_b be the two corresponding solutions to (1.3), for $a, b \in \mathcal{A}$. We use the notation

$$\delta := a - b, \qquad E := u_a - u_b$$

throughout the paper and we define the linear functional $L : H_0^1(D) \to \mathbb{R}$,

$$L(v) := \int_D \delta\nabla u_a \cdot \nabla v, \quad v \in H_0^1(D).$$

By subtracting the two weak equations (1.3) for a and b, we derive another representation of L,

$$L(v) = -\int_D b\nabla E \cdot \nabla v, \quad v \in H_0^1(D). \tag{2.8}$$

The following theorem gives two basic estimates for bounding the difference $\delta = a - b$. The first one illustrates that difficulties arise when $a - b$ changes sign, while the second puts forward the role of the weight $w = a|\nabla u_a|^2 + fu_a$.

Theorem 2.2. *Let D be a Lipschitz domain. Consider equation* (1.3) *with diffusion coefficients a and b. The following two inequalities hold for $\delta := a - b$.*

(i) *For any $a, b \in \mathcal{A}$ and $f \in H^{-1}(D)$, we have*

$$\left|\int_D \delta|\nabla u_a|^2\right| \leq \Lambda\|f\|_{H^{-1}(D)}\|E\|_{H_0^1(D)}.$$

(ii) *For any $a, b \in \mathcal{A}_1$ and $f \in L_\infty(D)$, we have*

$$\int_D \frac{\delta^2}{a^2}\Big(a|\nabla u_a|^2 + fu_a\Big) \leq C_0\|E\|_{H_0^1(D)}, \tag{2.9}$$

where

$$C_0 := C\|f\|_{L_\infty(D)}(1+\max\{\|\nabla a\|_{L_2(D)}, \|\nabla b\|_{L_2(D)}\}), \tag{2.10}$$

and C is a constant depending only on D, d, λ, Λ.

Proof: To prove (i), we take $v = u_a \in H_0^1(D)$ and obtain

$$L(u_a) = \int_D \delta|\nabla u_a|^2.$$

Using this in (2.8) yields

$$\int_D \delta|\nabla u_a|^2 = -\int_D b\nabla E\cdot\nabla u_a \le \Lambda\|u_a\|_{H_0^1(D)}\|E\|_{H_0^1(D)}. \tag{2.11}$$

If we take $v = -u_a$, we derive the same estimate for the negative of the left side of (2.11) which yields (i).

To prove (ii), we define $\bar\delta := \delta/a$ which belongs to $H^1(D)$ since $a, b \in \mathcal{A}_1$. Integrating by parts, we have for any $v \in H_0^1(D)$,

$$L(v) = \int_D \bar\delta a\nabla u_a\cdot\nabla v = -\int_D \nabla\bar\delta\cdot\nabla u_a a v - \int_D \bar\delta \operatorname{div}(a\nabla u_a)v. \tag{2.12}$$

Since $f = -\operatorname{div}(a\nabla u_a)$, this gives

$$L(v) = \frac{1}{2}\int_D \bar\delta a\nabla u_a\cdot\nabla v - \frac{1}{2}\int_D \nabla\bar\delta\cdot\nabla u_a a v + \frac{1}{2}\int_D \bar\delta f v, \quad v \in H_0^1(D). \tag{2.13}$$

Now, we chose $v = \bar\delta u_a \in H_0^1(D)$ to obtain

$$L(\bar\delta u_a) = \frac{1}{2}\int_D \bar\delta^2 a|\nabla u_a|^2 + \frac{1}{2}\int_D \bar\delta^2 f u_a. \tag{2.14}$$

Inserting (2.14) into (2.8) results in

$$\frac{1}{2}\int_D \bar\delta^2 a|\nabla u_a|^2 + \frac{1}{2}\int_D \bar\delta^2 f u_a = -\int_D b\nabla E\cdot\nabla(\bar\delta u_a) \le \Lambda\|\nabla(\bar\delta u_a)\|_{L_2(D)}\|E\|_{H_0^1(D)}. \tag{2.15}$$

Now, we resort to the estimate (see e.g. Chapter 8 in [12])

$$\|u_a\|_{L_\infty(D)} \le C\|f\|_{L_\infty(D)},$$

where C depends only on λ, Λ and D (throughout the rest of this proof $C > 0$ will be a generic constant that depends on at most d, D, λ, Λ). We use this result together with the energy estimate

$$\|\nabla u_a\|_{L_2(D)} \le \|f\|_{H^{-1}(D)} \le C\|f\|_{L_\infty(D)}$$

to obtain the bound

$$\begin{aligned}
\|\nabla(\bar{\delta}u_a)\|_{L_2(D)} &\le \left\|\frac{\delta}{a}\right\|_{L_\infty(D)} \|\nabla u_a\|_{L_2(D)} + \left\|\frac{u_a}{a}\right\|_{L_\infty(D)} \|\nabla\delta\|_{L_2(D)} + \left\|\frac{\delta}{a^2}\right\|_{L_\infty(D)} \|u_a\|_{L_\infty(D)} \|\nabla a\|_{L_2(D)} \\
&\le 2\Lambda\lambda^{-1}\|\nabla u_a\|_{L_2(D)} + \lambda^{-1}\|u_a\|_{L_\infty(D)}\|\nabla\delta\|_{L_2(D)} + 2\Lambda\lambda^{-2}\|u_a\|_{L_\infty(D)}\|\nabla a\|_{L_2(D)} \\
&\le C\|f\|_{L_\infty(D)}(1+\max\{\|\nabla a\|_{L_2(D)}, \|\nabla b\|_{L_2(D)}\}). \qquad (2.16)
\end{aligned}$$

Finally, plugging this estimate into (2.15), we derive that

$$\begin{aligned}
\int_D \frac{\delta^2}{a}|\nabla u_a|^2 + \int_D \frac{\delta^2}{a^2} f u_a &= \int_D \bar{\delta}^2 a|\nabla u_a|^2 + \int_D \bar{\delta}^2 f u_a \le 2\Lambda\|\nabla(\bar{\delta}u_a)\|_{L_2(D)}\|E\|_{H_0^1(D)} \\
&\le C\|f\|_{L_\infty(D)}(1+\max\{\|\nabla a\|_{L_2(D)}, \|\nabla b\|_{L_2(D)}\})\|E\|_{H_0^1(D)},
\end{aligned}$$

and the proof is completed. □

Note that when $a \le b$ or $b \le a$ a.e. on D and condition (1.11) holds in the sense that $\nabla u_a \cdot \tau \ge c > 0$, then part (i) gives the stability estimate

$$\|a-b\|_{L_1(D)} \le C\|f\|_{H^{-1}(D)}\|u_a - u_b\|_{H_0^1(D)}.$$

However, we can not claim such a result if the difference $(a-b)$ changes sign on a subset of D with a positive measure. In the sequel of the paper, we will not use (i), and instead rely only on (ii).

3 Improvements of Theorem 2.2

Theorem 2.2 is not satisfactory as it stands, since we want to replace the left side of (2.9), by $\|a-b\|^2_{L_2(D)}$. Obviously, this is possible when there exists a constant $c>0$ such that the weight satisfies

$$a|\nabla u_a|^2 + f u_a \ge c \quad \text{a.e. on} \quad D. \qquad (3.1)$$

In order to understand this condition, suppose that f does not change sign. In that case, the weak maximum principle [12] guarantees that u_a has the same sign as f and therefore the product $u_a f \ge 0$. Hence, (3.1) requires that u_a and $|\nabla u_a|$ do not vanish simultaneously. We prove in §3.1 that such a constant c exists provided certain (strong) smoothness assumptions for the diffusion coefficient a, the right side f, and the domain D hold. However, in order to allow milder regularity assumptions, we introduce the following weaker positivity condition.

Definition 3.1 (Positivity Condition)**.** *We say that (D,f,a) satisfy the positivity condition* $\mathbf{PC}(\beta)$ *if there exists a constant $c>0$ such that*

$$a(x)|\nabla u_a(x)|^2 + f(x)u_a(x) \ge c\,\mathrm{dist}(x,\partial D)^\beta, \quad \textit{a.e. } x \in D. \qquad (3.2)$$

Notice the positivity condition $\mathbf{PC}(0)$ is (3.1). In Lemma 3.7, we show that for every Lipschitz domain D and $a \in \mathcal{A}$, we have that (D,a,f) satisfies the positivity condition $\mathbf{PC}(2)$ provided f is strictly positive and in $L_2(D)$. In fact, in this case, the constant c in (3.2) is uniform over the class $\mathcal{A}$. In addition, we provide examples which show that additional regularity assumptions are required for (D,a,f) to satisfy the positivity condition $\mathbf{PC}(\beta)$ if $\beta < 2$. For now, we prove the following theorem which shows how a positivity condition $\mathbf{PC}(\beta)$ guarantees a stability estimate of the type we want.

Theorem 3.2. *Let D be a Lipschitz domain. Assume that a, $b \in \mathcal{A}_1$, $f \in L_\infty(D)$ and denote by u_a, u_b the corresponding solutions to* (1.3)*. If (D, a, f) satisfies the positivity condition* $\mathbf{PC}(\beta)$ *for $\beta \geq 0$, then we have*

$$\|a - b\|_{L_2(D)} \leq C\sqrt{1 + C_0}\|u_a - u_b\|_{H_0^1(D)}^{\frac{1}{2(\beta+1)}}, \tag{3.3}$$

where C_0 is the constant from (2.10) *and C is a constant depending only on D, d, λ, Λ, and c the constant in* (3.2)*.*

Proof: We recall the notation $\delta = a - b$, $E = u_a - u_b$, and start with the weighted L_2 estimate (2.9) provided in Theorem 2.2, namely

$$\int_D \frac{\delta^2}{a^2} w \leq C_0 \|E\|_{H_0^1(D)}, \quad w := a|\nabla u_a|^2 + f u_a, \tag{3.4}$$

where C_0 is the constant in (2.10). This proves the result in the case $\|E\|_{H_0^1(D)} = 0$ since $w > 0$ on D. Therefore, in going further, we assume $\|E\|_{H_0^1(D)} > 0$.

The presence of the non-negative weight w is handled by decomposing the domain D into two sets

$$D_\rho := \{x \in D \ : \ \operatorname{dist}(x, \partial D) \geq \rho\} \qquad \text{and} \qquad D_\rho^c := D \setminus D_\rho,$$

where $\rho > 0$ is to be chosen later. The triplet (D, a, f) satisfies the positivity condition $\mathbf{PC}(\beta)$, which guarantees that $w \geq c\rho^\beta$ on D_ρ. Hence, we deduce that

$$\int_{D_\rho} \delta^2 \leq \Lambda^2 c^{-1} \rho^{-\beta} \int_D \frac{\delta^2}{a^2} w \leq \Lambda^2 c^{-1} C_0 \rho^{-\beta} \|E\|_{H_0^1(D)}. \tag{3.5}$$

On D_ρ^c, the Lipschitz regularity assumption on ∂D implies the existence of a constant B such that $|D_\rho^c| \leq B\rho$. As a consequence, we obtain

$$\int_{D_\rho^c} \delta^2 \leq 4\Lambda^2 |D_\rho^c| \leq 4\Lambda^2 B\rho. \tag{3.6}$$

Combining the last two estimates with the choice $\rho = \|E\|_{H_0^1(D)}^{\frac{1}{\beta+1}}$ proves (3.3) and ends the proof. $\square$

3.1 The positivity condition $\mathbf{PC}(0)$

In view of the exponent in (3.3), the strongest stability occurs when $\beta = 0$. In this section, we show that if (D, a, f) are sufficiently smooth then $\mathbf{PC}(0)$ is satisfied. We denote by $C^{k,\alpha}(D)$, $k \in \mathbb{N}_0$, $0 < \alpha \leq 1$, the Hölder spaces equipped with the semi-norms

$$|f|_{C^{k,\alpha}(D)} := \sup_{|\gamma|=k} \sup_{x,y \in D,\, x \neq y} \left\{ \frac{|\partial^\gamma f(x) - \partial^\gamma f(y)|}{|x - y|^\alpha} \right\},$$

and norms

$$\|f\|_{C^{k,\alpha}(D)} := \sup_{|\gamma| \leq k} \|\partial^\gamma f\|_{L_\infty(D)} + |f|_{C^{k,\alpha}(D)}.$$

3.1.1 Sufficient conditions

The following lemma gives a sufficient condition for (D, a, f) to satisfy the positivity condition $\mathbf{PC}(0)$.

Lemma 3.3. *Assume that for some $\alpha > 0$, D is a $C^{2,\alpha}$ domain and $f \in C^{0,\alpha}(D)$ with $f \geq c_f > 0$. Furthermore, assume that the diffusion coefficient a belongs to $\mathcal{A} \cap C^{1,\alpha}(D)$, with*

$$\|a\|_{C^{1,\alpha}(D)} \leq A. \tag{3.7}$$

Then, the triplet (D, a, f) satisfies the positivity condition $\mathbf{PC}(0)$, with constant c depending on $D, \lambda, \Lambda, \|f\|_{C^{0,\alpha}}, c_f$ and A.

Proof: We have that

$$a(x)|\nabla u_a(x)|^2 + f(x)u_a(x) \geq \min\{\lambda, c_f\} \left(|\nabla u_a(x)|^2 + u_a(x)\right),$$

since $u_a \geq 0$ according to the weak maximum principle [12]. We proceed by showing that $|\nabla u_a|^2 + u_a \geq c$, a.e. on D. We do this by contradiction. Assume that there exists a sequence $\{a_n\}_{n\geq 0}$ of diffusion coefficients $a_n \in \mathcal{A}$ with $\|a_n\|_{C^{1,\alpha}(D)} \leq A$ such that, for each $n \geq 0$, there exists $x_n \in D$ with

$$|\nabla u_{a_n}(x_n)|^2 + u_{a_n}(x_n) \leq \frac{1}{n}. \tag{3.8}$$

Note that the assumptions of the theorem imply that the equation (1.3) holds in the strong sense. Then, the classical Schauder estimates, see [12], tell us that

$$\|u_{a_n}\|_{C^{2,\alpha}(D)} \leq C, \tag{3.9}$$

where C depends on A, D, α, λ and Λ. Then by compactness, up to a triple subsequence extraction, we may assume that

(i) a_n converge in C^1 towards a limit a^*,

(ii) u_{a_n} converges in C^2 towards a limit u^*,

(iii) x_n converges in $\overline{D}$ towards a limit x^*.

Therefore, the equation

$$-a^*\nabla u^* - \nabla a^* \cdot \nabla u^* = f, \tag{3.10}$$

is satisfied on D, with homogeneous boundary conditions, and we have

$$u^*(x^*) = 0 \quad \text{and} \quad \nabla u^*(x^*) = 0. \tag{3.11}$$

The first equality shows that x^* lies on the boundary, due to the strong maximum principle, and therefore the second equality contradicts the Hopf lemma, see [12]. $\square$

We have the following corollary.

Corollary 3.4. *Assume that for some $\alpha > 0$, D is a $C^{2,\alpha}$ domain, $f \in C^{0,\alpha}(D)$ with $f \geq c_f > 0$ and the diffusion coefficient $a \in \mathcal{A} \cap C^{1,\alpha}(D)$, with $\|a\|_{C^{1,\alpha}(D)} \leq A$. Furthermore, assume that $b \in \mathcal{A}_1$. Let u_a and u_b be the corresponding solutions to* (1.3)*, then*

$$\|a - b\|_{L_2(D)} \leq C_0 \|u_a - u_b\|^{1/2}_{H^1_0(D)}, \tag{3.12}$$

where $C_0 = C\|f\|^{1/2}_{L_\infty(D)}(1 + \max\{\|\nabla a\|_{L_2(D)}, \|\nabla b\|_{L_2(D)}\})^{1/2}$ and C is a constant depending only on $D, d, \lambda, \Lambda, c_f, \|f\|_{C^{0,\alpha}}$, and A. In particular, under the same assumptions on D, f, and b, we have the estimate

$$\|a - b\|_{L_2(D)} \leq C_s \|u_a - u_b\|^{1/2}_{H^1_0(D)}, \quad a \in \mathcal{A}_s, \tag{3.13}$$

for all $s > 1 + \frac{d}{2}$.

Proof: The inequality (3.12) follows from Theorem 3.2 and Lemma 3.3, while (3.13) follows by the Sobolev embedding of H^s into the relevant Hölder spaces. □

3.1.2 The condition PC(β), $\beta < 2$, requires smooth domains

In this section, we show that we cannot expect the triplet (D, a, f) to satisfy a positivity condition **PC**(β), $\beta < 2$, without additional regularity assumptions on the domain D. We consider the problem,

$$\begin{aligned} -\Delta u &= 1, \quad \text{on } D = (0,1)^d, \\ u &= 0, \quad \text{on } \partial D, \end{aligned} \tag{3.14}$$

corresponding to the case $a = 1$, $f = 1$, $D = (0,1)^d$. We begin with the following lemma.

Lemma 3.5. *The solution u to* (3.14) *is in the Hölder space $C^{1,\alpha}(D)$ for all $0 < \alpha < 1$.*

Proof: The solution u can be expanded in the eigenfunction basis

$$u(x) = \sum_{n \in \mathbb{N}^d} c_n s_n(x), \quad s_n(x) := \prod_{i=1}^{d} \sin\Big(\pi n_i x_i\Big), \quad x = (x_1, \dots, x_d), \tag{3.15}$$

with coefficients c_n, $n = (n_1, \dots, n_d)$, given by the formula

$$c_n = \begin{cases} \frac{4^d}{\pi^{2+d}(n_1^2 + \cdots + n_d^2) n_1 \dots n_d}, & \text{if all } n_i \text{ are odd}, \\ 0, & \text{otherwise}. \end{cases}$$

To prove the stated smoothness for the partial derivative $\frac{\partial u}{\partial x_1}$, we first show that

$$\sum_{n \in \mathbb{N}^d} \frac{1}{(n_1^2 + \cdots + n_d^2) n_2 \dots n_d} < \infty. \tag{3.16}$$

For this, we use the fact that, for any $A > 0$,

$$\sum_{k \geq 1} (A + k^2)^{-1} \leq \int_0^\infty (A + t^2)^{-1} dt = \frac{\pi}{2\sqrt{A}},$$

and thus

$$\sum_{n\in\mathbb{N}^d}\frac{1}{(n_1^2+\cdots+n_d^2)n_2\dots n_d} \le \frac{\pi}{2}\sum_{(n_2,\dots,n_d)\in\mathbb{N}^{d-1}}\frac{1}{n_2\dots n_d\sqrt{n_2^2+\cdots+n_d^2}}$$
$$\le \frac{\pi}{2(d-1)^{\frac12}}\sum_{(n_2,\dots,n_d)\in\mathbb{N}^{d-1}}\frac{1}{(n_2\dots n_d)^{1+\frac{1}{d-1}}}$$
$$= \frac{\pi}{2(d-1)^{\frac12}}\Big(\sum_{k\ge 1}k^{-1-\frac{1}{d-1}}\Big)^{d-1}<\infty,$$

where we have used the inequality between the arithmetic and geometric mean of $n_2^2,\dots,n_d^2$.

From (3.16), we can differentiate u termwise and obtain that $\frac{\partial u}{\partial x_1}$ is continuous. The same holds for all other partial derivatives, and thus $u\in C^1(D)$. In order to prove that u belongs to the Hölder space $C^{1,\alpha}(D)$ for sufficiently small $\alpha>0$, it suffices to check in addition that

$$\sum_{n\in\mathbb{N}^d}\frac{n_i^\alpha}{(n_1^2+\cdots+n_d^2)n_2\dots n_d}<\infty,\quad i=1,\dots,d.$$

Each term in this series is less than $\frac{1}{(n_1^2+\cdots+n_d^2)^{1-\frac{\alpha}{2}}n_2\dots n_d}$. We thus proceed to a similar computation using the fact that

$$\sum_{k\ge1}(A+k^2)^{-1+\frac{\alpha}{2}}\le\frac{C}{(\sqrt{A})^{1-\alpha}},$$

and derive that

$$\sum_{n\in\mathbb{N}^d}\frac{n_i^\alpha}{(n_1^2+\cdots+n_d^2)n_2\dots n_d}\le C\Big(\sum_{k\ge1}k^{-1-\frac{1-\alpha}{d-1}}\Big)^{d-1}<\infty,$$

since $\alpha<1$. □

The above lemma allows us to show that the positivity condition $\mathbf{PC}(\beta)$ does not hold for $\beta<2$, and in particular when $\beta=0$ when $D=(0,1)^d$.

Proposition 3.6. *Let $D=(0,1)^d$ and $a=f=1$, with $d\ge2$. Then the triplet (D,a,f) does not satisfy the positivity condition $\mathbf{PC}(\beta)$ if $\beta<2$.*

Proof: As shown in Lemma 3.5, the solution u to (3.14) is in the class $C^{1,\alpha}(D)$ for all $0<\alpha<1$, and therefore ∇u can be continuously extended up to the boundary ∂D. Since the tangential derivatives of u vanish on the boundary, it follows that when x^* is a corner of the cube $[0,1]^d$, then $\nabla u(x^*)=0$. By Hölder regularity, we find that

$$|\nabla u(x)|\le C\mathrm{dist}(x,x^*)^\alpha\quad\text{and}\quad|u(x)|\le C\mathrm{dist}(x,x^*)^{1+\alpha},\quad x\in D,\tag{3.17}$$

and therefore

$$a(x)|\nabla u_a(x)|^2+f(x)u_a(x)\le C\mathrm{dist}(x,x^*)^{2\alpha},\quad x\in D,\tag{3.18}$$

for all $0<\alpha<1$. Thus, $\mathbf{PC}(\beta)$ cannot hold for any $\beta<2$. □

3.2 The positivity condition $\mathbf{PC}(2)$

In this section, we show that the triplet (D, a, f) satisfies the positivity condition $\mathbf{PC}(2)$ for any Lipschitz domain D, any $a \in \mathcal{A}$, and any $f \in L_2(D)$, with $f \geq c_f > 0$. For this, we use the lower bounds on the Green functions established in [13].

Lemma 3.7. *Let D be a Lipschitz domain, $a \in \mathcal{A}$, and $f \in L_2(D)$ with $f \geq c_f > 0$. Then the triplet (D, a, f) satisfies the positivity condition $\mathbf{PC}(2)$ with a constant c only depending on $\lambda, \Lambda, d, D, c_f$.*

Proof: In this proof, C denotes a generic constant only depending on $D, \lambda, \Lambda, d, c_f$. We recall that for every $y \in D$, there exists a unique Green's function $G_a(\cdot, y) \in W_0^1(L_1(D))$, such that

$$\int_D \nabla G_a(x,y) \nabla v(x)\, dx = v(y), \quad v \in C_0^\infty(D).$$

One can show that

$$G_a(x,y) \geq C|x-y|^{-(d-2)}, \quad \text{for} \quad |x-y| \leq \frac{1}{2}\rho(x), \quad d \geq 2,$$

where $\rho(x) := \operatorname{dist}(x, \partial D)$. A proof of this fact in the case $d \geq 3$ can be found in [13, Theorem 1.1]. The same proof holds also in the case $d = 2$, utilizing the regularity properties of the two dimensional Green's function discussed in [7].

Now, given any $x \in D$, let $B(x, \rho(x)/2) \subset D$ be the ball centered at x with radius $\rho(x)/2$. Since $G_a(x,y) \geq 0$, $x, y \in D$, we have

$$\begin{aligned} u_a(x) &= \int_D f(y) G_a(x,y)\, dy \geq \int_B f(y) G_a(x,y)\, dy \\ &\geq C \int_{B(x,\rho(x)/2)} |x-y|^{-(d-2)}\, dy \geq C\rho^2(x) = C[\operatorname{dist}(x,\partial D)]^2, \end{aligned}$$

and the desired result follows. □

We have the following corollary.

Corollary 3.8. *Let D be a Lipschitz domain, $a, b \in \mathcal{A}_1$, $f \in L_\infty(D)$ with $f \geq c_f > 0$, and $u_a, u_b \in H_0^1(D)$ be the corresponding solutions to (1.3), then we have*

$$\|a - b\|_{L_2(D)} \leq C\sqrt{1 + C_0}\,\|u_a - u_b\|_{H_0^1(D)}^{1/6}, \tag{3.19}$$

where C_0 is the constant in (2.10) and C is a constant depending only on D, d, λ, Λ and the minimum c_f of f.

Proof: The proof follows from Theorem 3.2 and Lemma 3.7. □

4 Finer estimates for parameter recovery

We have proved Corollary 3.8 for Lipschitz domains D under the assumptions that $a, b \in \mathcal{A}_1$ and $f \in L_\infty(D)$, with $f \geq c_f > 0$. In this section, we shall weaken the smoothness assumption on a and b at the expense of decreasing the exponent $1/6$ appearing on the right side of (3.19).

4.1 Finer estimates

Our method for reducing the smoothness assumptions on the diffusion coefficients in the stability Theorem 3.2 will be based on interpolation. We recall that if $a \in H^s(D)$, where $D \subset \mathbb{R}^d$ is a bounded Lipschitz domain, then for each $t > 0$, there is a function $a_t \in H^1(D)$ satisfying the inequality

$$\|a - a_t\|_{L_2(D)} + t\|\nabla a_t\|_{L_2(D)} \leq Ct^s\|a\|_{H^s(D)}, \tag{4.1}$$

where the constant C depends only on D. Note that the standard construction of a_t is a local mollification of a, and therefore $a_t \in \mathcal{A}$ whenever $a \in \mathcal{A}$.

Our stability estimate relies on the following result which can be derived from Theorem 2.1 in [4]:

Lemma 4.1. *Given $a, b \in \mathcal{A}$, assume that for some $0 < \theta \leq 1$ there exists a constant M such that*

$$\|\nabla u_a\|_{L_{2/(1-\theta)}(D)} \leq M.$$

Then,

$$\|u_a - u_b\|_{H_0^1(D)} \leq \lambda^{-1}(2\Lambda)^{1-\theta}M\|a - b\|_{L_2(D)}^{\theta}. \tag{4.2}$$

Proof: We take $p = \frac{2}{1-\theta}$ in Theorem 2.1 of [4], then for $q = \frac{2}{\theta}$, we have from (2.2) of [4]

$$\|u_a - u_b\|_{H_0^1(D)} \leq \lambda^{-1}M\|a - b\|_{L_q(D)}^{\theta}. \tag{4.3}$$

Since $\|a - b\|_{L_q(D)} \leq \|a - b\|_{L_2(D)}^{2/q}(2\Lambda)^{1-2/q}$, the lemma follows. □

This motivates the following definition.

Definition 4.2 (Gradient Condition)**.** *We say that a function $u \in H_0^1(D)$ satisfies the gradient condition* $\mathbf{GC}(\theta, M)$*, $0 < \theta \leq 1$, if*

$$\|\nabla u\|_{L_{2/(1-\theta)}(D)} \leq M. \tag{4.4}$$

We now prove our main result regarding stable recovery of parameters provided that u_a satisfies the gradient condition $\mathbf{GC}(\theta, M)$. Later, in §4.2, we elaborate on what classical smoothness conditions on the diffusion coefficient $a \in \mathcal{A}$ guarantees that this gradient condition holds.

Theorem 4.3. *Let D be a Lipschitz domain, $f \in L_\infty(D)$ with $f \geq c_f > 0$, and $a, b \in \mathcal{A}_s$ for some $1/2 < s \leq 1$. Let $u_a, u_b \in H_0^1(D)$ be the corresponding solutions to* (1.3)*. If u_a, u_b both satisfy the gradient condition* $\mathbf{GC}(\theta, M)$ *for some $\frac{1-s}{s} < \theta \leq 1$, then we have*

$$\|a - b\|_{L_2(D)} \leq C\sqrt{1 + (\|a\|_{H^s(D)} + \|b\|_{H^s(D)})^{\frac{1}{3s}}}\|u_a - u_b\|_{H_0^1(D)}^{\frac{1}{6} - \frac{1-s}{6s\theta}}, \tag{4.5}$$

where C is a constant depending only on $D, d, \theta, \lambda, \Lambda$, the minimum c_f of f, $\|f\|_{L_\infty(D)}$, and M.

Proof: We use the notation

$$E := u_a - u_b, \quad E_t := u_{a_t} - u_{b_t}, \quad \delta := a - b, \quad \delta_t := a_t - b_t,$$

where $a_t, b_t \in \mathcal{A}_1$ are the functions satisfying (4.1). Throughout the proof $C > 0$ will be a generic constant that depends on at most $D, d, \theta, \lambda, \Lambda$, M, $\|f\|_{L_\infty(D)}$, and the minimum c_f of f. In what follows, the value of C may change at each appearance. We denote by

$$M_0 := \|a\|_{H^s(D)} + \|b\|_{H^s(D)} \geq \|a\|_{L_2(D)} + \|b\|_{L_2(D)} \geq 2\lambda |D|^{1/2}. \tag{4.6}$$

It follows from (4.1) that

$$\|\delta - \delta_t\|_{L_2(D)} \leq C M_0 t^s. \tag{4.7}$$

We want to bound $\|\delta\|_{L_2(D)}$. For this, we define the set $D_\rho := \{x \in D \ : \ \mathrm{dist}(x, \partial D) \geq \rho\}$, with the value of $\rho > 0$ to be chosen shortly. Using (4.7), we find that

$$\begin{aligned} \|\delta\|^2_{L_2(D)} &= \|\delta\|^2_{L_2(D^c_\rho)} + \|\delta\|^2_{L_2(D_\rho)} \leq \|\delta\|^2_{L_2(D^c_\rho)} + 2\|\delta - \delta_t\|^2_{L_2(D)} + 2\|\delta_t\|^2_{L_2(D_\rho)} \\ &\leq \|\delta\|^2_{L_2(D^c_\rho)} + C M_0^2 t^{2s} + 2\|\delta_t\|^2_{L_2(D_\rho)}. \end{aligned} \tag{4.8}$$

To estimate the two norms above, we proceed as in the proof of Theorem 3.2. First, for a, $b \in \mathcal{A}$ and a Lipschitz domain D we have

$$\|\delta\|^2_{L_2(D^c_\rho)} = \int_{D^c_\rho} \delta^2 \leq 4\Lambda^2 |D^c_\rho| \leq C\rho; \tag{4.9}$$

see (3.6). Since a_t and b_t are in $\mathcal{A}_1$, according to Lemma 3.7, (D, a_t, f) and (D, b_t, f) satisfiy the positivity condition $\mathbf{PC}(2)$ with a constant c only depending on λ, Λ, D, d. Hence (3.5) holds with $\beta = 2$ and therefore, we have

$$\|\delta_t\|^2_{L_2(D_\rho)} = \int_{D_\rho} \delta_t^2 \leq C\rho^{-2}(1 + \max\{\|\nabla a_t\|_{L_2(D)}, \|\nabla b_t\|_{L_2(D)}\})\|E_t\|_{H^1_0(D)}.$$

This, together with (4.1) implies that

$$\|\delta_t\|^2_{L_2(D_\rho)} \leq C\rho^{-2}(1 + M_0 t^{s-1})\|E_t\|_{H^1_0(D)}. \tag{4.10}$$

We substitute (4.9) and (4.10) into (4.8) to arrive at

$$\|\delta\|^2_{L_2(D)} \leq C\rho + C M_0^2 t^{2s} + C\rho^{-2}(1 + M_0 t^{s-1})\|E_t\|_{H^1_0(D)}. \tag{4.11}$$

We now proceed to estimate $\|E_t\|_{H^1_0(D)}$ by taking advantage of the gradient condition $\mathbf{GC}(\theta, M)$ satisfied by u_a and u_b. Since u_a satisfies the gradient condition $\mathbf{GC}(\theta, M)$ and $a_t \in \mathcal{A}$, it follows from the stability estimate (4.2) that

$$\|u_a - u_{a_t}\|_{H^1_0(D)} \leq C\|a - a_t\|^\theta_{L_2(D)} \leq C(M_0 t^s)^\theta. \tag{4.12}$$

The same estimate holds with a replaced by b, and therefore

$$\|E_t\|_{H^1_0(D)} \leq \|u_{a_t} - u_a\|_{H^1_0(D)} + \|u_a - u_b\|_{H^1_0(D)} + \|u_b - u_{b_t}\|_{H^1_0(D)} \leq C(M_0 t^s)^\theta + \|E\|_{H^1_0(D)}. \tag{4.13}$$

Placing this estimate into (4.11) gives

$$\|\delta\|^2_{L_2(D)} \leq C\rho + C M_0^2 t^{2s} + C\rho^{-2}(1 + M_0 t^{s-1})(M_0^\theta t^{s\theta} + \|E\|_{H^1_0(D)}). \tag{4.14}$$

To finish the proof, we consider two cases.

Case 1: $\|E\|_{H_0^1(D)} > 0$. First, we choose t so that $M_0^\theta t^{s\theta} = \|E\|_{H_0^1(D)}$, i.e. $t := \|E\|_{H_0^1(D)}^{\frac{1}{s\theta}} M_0^{-1/s}$, so that the two terms in the last bracketed sum of (4.14) are equal. Since

$$\|E\|_{H_0^1(D)} \leq C, \tag{4.15}$$

and $M_0 \geq C$ (because of (4.6)), this choice of t satisfies

$$1 \leq C M_0 t^{s-1}. \tag{4.16}$$

Next, we choose ρ such that $\rho^3 = M_0 t^{s-1} \|E\|_{H_0^1(D)} = M_0^{1/s} \|E\|_{H_0^1(D)}^{\frac{s\theta+s-1}{s\theta}}$. This choice balances the first and last terms on the right side of (4.14) and therefore gives

$$\|\delta\|_{L_2(D)}^2 \leq C M_0^{\frac{1}{3s}} \|E\|_{H_0^1(D)}^{\frac{s\theta+s-1}{3s\theta}} + C\|E\|_{H_0^1(D)}^{\frac{2}{\theta}}. \tag{4.17}$$

Since $\frac{s\theta+s-1}{3s} \leq 2$, the inequalities (4.15) and (4.16) show that the first term in the sum on the right can be absorbed into the second, and the theorem follows.

Case 2: $\|E\|_{H_0^1(D)} = 0$. For any sufficiently small $t > 0$, we choose ρ such that $\rho^3 = M_0^{1+\theta} t^{s\theta+s-1}$ so that the first and last terms in (4.14) balance. Then, (4.14) gives

$$\|\delta\|_{L_2(D)}^2 \leq C M_0^{\frac{1+\theta}{3}} t^{\frac{s\theta+s-1}{3}} + C M_0^2 t^{2s}.$$

Since by assumption, $\theta > \frac{1-s}{s}$, we have $t^{s\theta+s-1} \to 0$ as $t \to 0$, and therefore (4.5) holds in this case as well. $\square$

Note that the proof of the above theorem relies on the fact that (D, a_t, f) and (D, b_t, f) both satisfy the positivity condition $\mathbf{PC}(2)$ for a uniform constant c. The proof can be easily modified to cover the case where (D, a_t, f) and (D, b_t, f) satisfy the positivity condition $\mathbf{PC}(\beta)$ with a uniform constant c for any given $0 \leq \beta < 2$.

Remark 4.4. *As noted in the introduction, a typical result based on least squares or variational techniques for finding the diffusion coefficient a is estimate* (1.12). *For clarity, we focus here on the results from [16, 9], where the approximation $a_h \in A_h$ is computed solely based on the knowledge of u^{ob}. Therefore any two diffusion coefficients a and b with the same observed u^{ob} will have the same approximant a_h, generated by the above process. If we take $u^{ob} = u_a$ in* (1.12), *we obtain the bound*

$$\|a - a_h\|_{L_2(D)} \leq C h^r. \tag{4.18}$$

On the other hand, we can view $u_a = u^{ob}$ as an observation of u_b and in this case obtain from (1.12), *the bound*

$$\|b - a_h\|_{L_2(D)} \leq C(h^r + h^{-2}\|u_a - u_b\|_{L_2(D)}). \tag{4.19}$$

Hence,

$$\|b - a\|_{L_2(D)} \leq C h^r + C(h^r + h^{-2}\|u_a - u_b\|_{L_2(D)}). \tag{4.20}$$

If we chose h, such that $h^r = h^{-2}\|u_a - u_b\|_{L_2(D)}$, we obtain the estimate

$$\|b - a\|_{L_2(D)} \leq C\|u_a - u_b\|_{L_2(D)}^{\frac{r}{r+2}}. \tag{4.21}$$

Besides working with Neumann boundary conditions, there are two major distinctions between (4.21) *and our results. The first is the* $L_2(D)$ *norm that appears on the right side in place of our* $H_0^1(D)$ *norm. Recall that we have already mentioned (see* (1.18)*) how one can derive bounds of the form* (4.21) *from our results. The second distinction is the much more demanding regularity assumption placed on* a, b *as well as on* u_a, u_b*. Namely,* (4.21) *is proved in the above references under the regularity requirements* $a, b \in H^{r+1}(D)$ *and* $u_a, u_b \in W^{r+3}(L_\infty(D))$ *with* $r \geq 1$*. Whereas, in our treatment, stability estimates are available solely under the much weaker stability assumption* $a, b \in H^s(D)$*,* $s* < s \leq 1$*, where* $s^* < 1$*.*

4.2 The gradient condition $\mathbf{GC}(\theta, M)$

The statement of Theorem 4.3 relies on the assumption that the solutions u_a and u_b satisfy the gradient condition $\mathbf{GC}(\theta, M)$. Finding sufficient conditions that ensure $\mathbf{GC}(\theta, M)$ is a well studied question in harmonic analysis and partial differential equations. We recall, two classes of diffusion coefficient for which such condition holds.

4.2.1 VMO diffusion coefficients

We start with the following result from [3].

Result 1. *If* D *is a* C^1 *domain, the diffusion coefficient* a *is in* $\mathrm{VMO} \cap \mathcal{A}$*, and the right side* $f = \mathrm{div}(g)$*, with* $g \in L_p(D)$*, then there exists a unique weak solution* u_a *to* (1.3) *such that* $\nabla u_a \in L_p(D)$*,* $1 < p < \infty$*, and*

$$\|\nabla u_a\|_{L_p(D)} \leq C\|g\|_{L_p(D)}, \tag{4.22}$$

with C *depending only on* $D, d, p, \lambda, \Lambda$ *and the VMO modulus of* a*.*

Recall that the VMO modulus $\nu(a, \cdot)$ of a is defined by

$$\nu(a, t) := \sup_{|Q| \leq t} \frac{1}{|Q|} \int_Q |a - a_Q|, \quad a_Q := \frac{1}{|Q|} \int_Q a, \quad t > 0,$$

where the supremum is taken over all cubes Q with measure at most t. In order to show that u_a satisfies the gradient condition $\mathbf{GC}(\theta, M)$, we need to consider a subclass of diffusion coefficients a, for which the estimate (4.22) is uniform for all functions in this class. For this, we consider a non-decreasing continuous function $\Phi(t)$, $t \geq 0$, with $\Phi(0) = 0$, and introduce the class $\mathcal{A}_\Phi$ defined as

$$\mathcal{A}_\Phi := \{a \in \mathcal{A} \, : \, \nu(a, t) \leq \Phi(t), \; t > 0\}. \tag{4.23}$$

Likewise, for $s > 0$, we define the class

$$\mathcal{A}_{s,\Phi} := \mathcal{A}_s \cap \mathcal{A}_\Phi. \tag{4.24}$$

An examination of the proofs in [3] and [10] shows that for all $a \in \mathcal{A}_\Phi$ the constant in (4.22) is uniformly bounded, with a bound, depending on Φ, D, d, λ, Λ. Therefore, according to the estimate (4.22), for each $0 < \theta < 1$, the solution u_a satisfies the gradient condition $\mathbf{GC}(\theta, M)$ with M only depending on θ, D, d, λ, Λ, Φ, and f. As a consequence, we deduce the following corollary of Theorem 4.3.

Corollary 4.5. *Let D be a C^1 domain, $f \in L_\infty(D)$ with $f \geq c_f > 0$ and $\Phi(t)$, $t \geq 0$, be a non-decreasing continuous function with $\Phi(0) = 0$. Furthermore, assume that $a, b \in \mathcal{A}_{s,\Phi}$ for some $\frac{1}{2} < s \leq 1$. Then there exists a constant C only depending on D, d, λ, Λ, f, and Φ such that*

$$\|a - b\|_{L_2(D)} \leq C\sqrt{1 + (\|a\|_{H^s(D)} + \|b\|_{H^s(D)})^{\frac{1}{3s}}}\|u_a - u_b\|^r_{H^1_0(D)} \tag{4.25}$$

for every $r < \frac{2s-1}{6s}$.

□

4.2.2 General diffusion coefficients

Again, we start with the following gradient estimate.

Result 2 (see [21, 4])**.** *If D is any Lipschitz domain, then there is a value $P > 2$, depending on D, such that whenever $a \in \mathcal{A}$ and $f \in W^{-1}(L_p(D))$, with $2 \leq p < P$, then*

$$\|\nabla u_a\|_{L_p(D)} \leq C\|f\|_{W^{-1}(L_p(D))},$$

with C depending only on $d, D, \lambda, \Lambda, p$.

It follows from the above result that u_a satisfies condition $\mathbf{GC}(\theta, M)$ for $0 < \theta < \frac{P-2}{P}$, where M depends on d, D, λ, Λ, and f. Therefore, Result 2 and Theorem 4.3 lead to the following corollary.

Corollary 4.6. *Let D be a Lipschitz domain, $f \in L_\infty(D)$ with $f \geq c_f > 0$ and let $P > 2$ be the constant in Result 2. Assume that $a, b \in \mathcal{A}_s$ with $\frac{P}{2(P-1)} < s \leq 1$. Then, there exists a constant C only depending on D, d, s, λ, Λ, and f such that*

$$\|a - b\|_{L_2(D)} \leq C\sqrt{1 + (\|a\|_{H^s(D)} + \|b\|_{H^s(D)})^{\frac{1}{3s}}}\|u_a - u_b\|^r_{H^1_0(D)}, \tag{4.26}$$

for every $r < \frac{1}{6} - \frac{P(1-s)}{6(P-2)s}$.

5 Piecewise constant diffusion coefficients

Piecewise constant diffusion coefficients are often used in numerical simulation. This case is not covered by the discussions in the preceding sections because such diffusion coefficients do not satisfy the regularity assumptions considered there. In this section, we derive some elementary results for piecewise constant parameters a, subordinate to a fixed partition. We assume for simplicity that the domain $D = (0,1)^d$ and $\mathcal{P}_n$ is the partition of D into n^d disjoint cubes of side length $1/n$. The derivations that follow can be generalized to other settings. We denote by $\mathcal{A}^n$ the set of all diffusion coefficients a defined on D that are piecewise constant functions subordinate to $\mathcal{P}_n$. We continue to make the assumption that each $a \in \mathcal{A}^n$ satisfies $\lambda \leq a \leq \Lambda$ for fixed $0 < \lambda < \Lambda$, and therefore can be written as

$$a := \sum_{Q \in \mathcal{P}_n} a_Q \chi_Q, \tag{5.1}$$

where $a_Q \in [\lambda, \Lambda]$, and χ_Q is the characteristic function of the cube Q.

Lemma 5.1. *Let $D = (0,1)^d$ and $f \in L_2(D)$. If the diffusion coefficient $a \in \mathcal{A}^n$ is given by* (5.1)*, then for each cube $Q \in \mathcal{P}_n$, the solution u_a to* (1.3) *satisfies the equation*

$$-a_Q \Delta u_a(x) = f(x), \quad a.e.\ x \in Q. \tag{5.2}$$

Proof: Let $a \in \mathcal{A}_n$ and $Q \in \mathcal{P}_n$. Following the proof of the interior regularity theorem, see [8], one can show that $u_a \in W^2(L_2(\mathcal{O}))$ on each open set $\mathcal{O}$ strictly contained in Q. If in (1.3), we take v smooth and compactly supported on Q and integrate by parts, we find

$$-a_Q \int_Q \Delta u_a v = \int_Q f v. \tag{5.3}$$

It follows that $-a_Q \Delta u_a = f$ at every point x in the interior of Q which is a Lebesgue point of both f and Δu_a. In particular, this holds almost everywhere on Q. □

Theorem 5.2. *Let $D = (0,1)^d$ and $f \in L_2(D)$ with $f \geq c_f > 0$ on D. Let $a, b \in \mathcal{A}^n$ be diffusion coefficients and u_a, u_b be the corresponding solutions to* (1.3) *on D. Then for each $Q \in \mathcal{P}_n$, we have*

$$|a_Q - b_Q| \leq C n^{\frac{d+2}{2}} \|\nabla u_a - \nabla u_b\|_{L_2(Q)}, \tag{5.4}$$

where C depends only on c_f and Λ. Therefore,

$$\|a - b\|_{L_2(D)} \leq C n \|u_a - u_b\|_{H_0^1(D)}. \tag{5.5}$$

Proof: From Lemma 5.1, we know that for each $Q \in \mathcal{P}_n$, we have

$$a_Q - b_Q = \Delta(u_a - u_b) \frac{a_Q b_Q}{f}, \quad \text{a.e. on } Q. \tag{5.6}$$

We now assume without loss of generality that $a_Q > b_Q$. Therefore, we have that $\Delta(u_a - u_b) > 0$ on Q since $f > 0$. Recall that there exist functions $\varphi_Q \in C_c^\infty(Q)$ (for example the standard mollifier supported in Q), such that $\int_Q \varphi_Q = 1$ and

$$\|\nabla \varphi_Q\|_{L_2(Q)} \leq C_0 n^{\frac{d+2}{2}}, \tag{5.7}$$

with C_0 an absolute constant. Then multiplying (5.6) by such a φ_Q and integrating over Q yields

$$a_Q - b_Q = \int_Q \Delta(u_a - u_b) \frac{a_Q b_Q}{f} \varphi_Q \leq \frac{a_Q b_Q}{c_f} \int_Q \Delta(u_a - u_b) \varphi_Q = -\frac{a_Q b_Q}{c_f} \int_Q \nabla(u_a - u_b) \nabla \varphi_Q,$$

where we used integration by parts to get the last equality. The boundedness of a and b yields

$$\begin{aligned} a_Q - b_Q &\leq C \|\nabla(u_a - u_b)\|_{L_2(Q)} \|\nabla \varphi_Q\|_{L_2(Q)} \\ &\leq C n^{\frac{d+2}{2}} \|\nabla(u_a - u_b)\|_{L_2(Q)}. \end{aligned} \tag{5.8}$$

This proves (5.4). To prove (5.5), we square (5.4) integrate over Q to find

$$\int_Q |a - b|^2 \leq C n^{d+2} \|\nabla(u_a - u_b)\|_{L_2(Q)}^2 n^{-d} = C n^2 \|\nabla(u_a - u_b)\|_{L_2(Q)}^2. \tag{5.9}$$

If we add these estimates up over all $Q \in \mathcal{P}_n$ and take a square root, we arrive at (5.5). □

6 The univariate case

In the univariate case, several stability results, mainly for the Neumann problem, are available, see for example, [19]. Here, we will discuss the one dimensional Dirichlet problem with diffusion coefficients $a \in \mathcal{A}$ and the domain $D = (0,1)$. In this case, under certain assumptions on f, we will be able to improve the Lipschitz exponent in the inverse parameter estimate and also provide limits to how large this Lipschitz exponent can be.

Notice that in this case, one needs some assumptions on f to guarantee that a is uniquely determined from the solution u_a, as the following example, taken from [19], shows. The function

$$u(x) = \begin{cases} x, & x \in [0, \frac{1}{2}], \\ 1 - x, & x \in (\frac{1}{2}, 1], \end{cases}$$

is a solution on D to the problem

$$-(au')' = 2\delta_{1/2}, \quad u(0) = u(1) = 0,$$

with diffusion coefficient $a \equiv 1$ or any a of the form

$$a = \begin{cases} q, & \text{on } [0, \frac{1}{2}], \\ 2 - q, & \text{on } (\frac{1}{2}, 1], \end{cases}$$

where $0 < q < 2$. Here $\delta_{1/2}$ is the delta distribution with weight 1 at 1/2.

In going further, we consider the case $f = 1$, noting that the derivations below can be generalized to other settings. We determine the solution u_a and show that estimate (3.19) in Corollary 3.8 can be improved. We use the notation $A := 1/a$, $B := 1/b$, where $a, b \in \mathcal{A}$. Now, (1.3) becomes

$$\int_0^1 a u_a' v' = \int_0^1 v, \quad v \in H_0^1(0,1), \tag{6.1}$$

and one checks that the solution to (6.1) is

$$u_a(x) = -\int_0^x A(t)(t - \gamma_a)\,dt, \quad \text{where} \quad \gamma_a := \frac{\int_0^1 A(t)t\,dt}{\int_0^1 A(t)\,dt} \in (0,1). \tag{6.2}$$

This gives

$$-A(x)(x - \gamma_a) = u_a'(x). \tag{6.3}$$

6.1 An upper bound

To bound $\|a - b\|_{L_2(0,1)}$ in terms of $\|u_a' - u_b'\|_{L_2(0,1)}$, it is sufficient to bound $\|A - B\|_{L_2(0,1)}$. Let us set $\eta := \gamma_a - \gamma_b$, and $E'(x) := u_a'(x) - u_b'(x)$. Without loss of generality, we may assume that $\eta \geq 0$, since otherwise we can reverse the roles of a and b. The following lemma gives an estimate for η.

Lemma 6.1. *We have*

$$\eta \leq c_0 \|E'\|_{L_2(0,1)}^{2/3}, \tag{6.4}$$

where the constant c_0 depends only on λ and Λ.

Proof: The estimate obviously holds if $\eta = 0$, so we assume that $\eta > 0$. We consider an interval I of length $2c\eta$ centered at γ_a with $c := \frac{\lambda}{2(\lambda+\Lambda)} < 1/2$. We have for $x \in I \cap (0,1)$

$$\begin{aligned} |u_a'(x) - u_b'(x)| &= |(x-\gamma_b)B(x) - (x-\gamma_a)A(x)| \geq (1-c)B(x)\eta - cA(x)\eta \\ &\geq \left(\frac{1-c}{\Lambda} - \frac{c}{\lambda}\right)\eta = \frac{\eta}{2\Lambda}. \end{aligned}$$

Squaring this estimate and integrating over $I \cap (0,1)$ gives

$$\frac{\eta^2}{4\Lambda^2}|I \cap (0,1)| \leq \|u_a' - u_b'\|_{L_2(0,1)}^2 = \|E'\|_{L_2(0,1)}^2,$$

and since $|I \cap (0,1)| \geq c\eta$, the proof is completed. $\square$

The following lemma gives an upper bound for the norm $\|A - B\|_{L_2(0,1)}$.

Lemma 6.2. *For every $\rho > 0$, we have*

$$\|A - B\|_{L_2(0,1)}^2 \leq \frac{C}{\rho^2}\|E'\|_{L_2(0,1)}^{4/3} + 8\lambda^{-2}\rho, \tag{6.5}$$

where C depends only on λ and Λ. In particular, if $\|E'\|_{L_2(0,1)} = 0$, then $A = B$ a.e in $(0,1)$.

Proof: First, let us observe that

$$(A(x)-B(x))(x-\gamma_a) = A(x)(x-\gamma_a)-B(x)(x-\gamma_b)+B(x)(\gamma_a-\gamma_b) = -E'(x)+B(x)(\gamma_a-\gamma_b). \tag{6.6}$$

We now consider an interval J of length 2ρ centered at γ_a. Then, using (6.6) on J^c, where J^c is the complement of J in $(0,1)$ (which might be empty), we have

$$\rho|(A(x) - B(x)| \leq |E'(x)| + \lambda^{-1}\eta, \quad x \in J^c,$$

and therefore

$$\rho^2|(A(x) - B(x)|^2 \leq 2|E'(x)|^2 + 2\lambda^{-2}\eta^2, \quad x \in J^c.$$

We integrate the latter inequality over J^c to obtain

$$\|A - B\|_{L_2(J^c)}^2 \leq \frac{2}{\rho^2}\|E'\|_{L_2(0,1)}^2 + \frac{2\lambda^{-2}}{\rho^2}\eta^2. \tag{6.7}$$

Meanwhile, for $x \in J \cap (0,1)$, we have $|A(x) - B(x)| \leq 2\lambda^{-1}$ and therefore

$$\|A - B\|_{L_2(J\cap(0,1))}^2 \leq 8\lambda^{-2}\rho. \tag{6.8}$$

Combining this with (6.7), we obtain

$$\begin{aligned} \|A - B\|_{L_2(0,1)}^2 &\leq \frac{2}{\rho^2}\|E'\|_{L_2(0,1)}^2 + \frac{2\lambda^{-2}}{\rho^2}\eta^2 + 8\lambda^{-2}\rho \\ &\leq \frac{2}{\rho^2}\|E'\|_{L_2(0,1)}^2 + \frac{2c_0^2}{\rho^2\lambda^2}\|E'\|_{L_2(0,1)}^{4/3} + 8\lambda^{-2}\rho, \end{aligned}$$

where we have used Lemma 6.1. Since $|u_a'(x) - u_b'(x)| = |(x-\gamma_b)B(x) - (x-\gamma_a)A(x)| \le 2\lambda^{-1}$, we have that $\|E'\|_{L_2(0,1)} \le 2\lambda^{-1}$, and the first term of the above inequality is absorbed by the second term. Hence, we get

$$\|A-B\|^2_{L_2(0,1)} \le \frac{C}{\rho^2}\|E'\|^{4/3}_{L_2(0,1)} + 8\lambda^{-2}\rho,$$

where C depends only on λ and Λ. This proves the first part of the lemma. When $\|E'\|_{L_2(0,1)} = 0$,

$$\|A-B\|^2_{L_2(0,1)} \le 8\lambda^{-2}\rho,$$

for all $\rho > 0$ and so $A = B$ a.e. in $(0,1)$. □

We can now prove the following stability estimate in the one dimensional case.

Theorem 6.3. *For any $a, b \in \mathcal{A}$, the solutions u_a, u_b to (1.3) with $f = 1$ satisfy the estimate*

$$\|a-b\|_{L_2(0,1)} \le C\|u_a - u_b\|^{2/9}_{H_0^1(0,1)}, \tag{6.9}$$

where C depends only on λ and Λ. In particular, if $u_a = u_b$ on $(0,1)$, then $a = b$ a.e in $(0,1)$.

Proof: If $\|u_a - u_b\|_{H_0^1(0,1)} = 0$, it follows from Lemma 6.2 that $a = b$, a.e. on $(0,1)$, and therefore (6.9) holds. When $\|E'\|_{L_2(0,1)} = \|u_a' - u_b'\|_{L_2(0,1)} > 0$, we choose $\rho = \|E'\|^{4/9}_{L_2(0,1)}$ in Lemma 6.2 to derive the desired estimate. □

6.2 A lower bound

In this section, we show that the exponent in estimates of the form (6.9) cannot be greater than 1/3.

Theorem 6.4. *Consider equation (1.3) with domain $D = (0,1)$ and right side $f = 1$. There are diffusion coefficients $a, b \in \mathcal{A}$, such that the corresponding solutions u_a, u_b, satisfy the inequality*

$$\|a-b\|_{L_2(D)} \ge c\|u_a - u_b\|^{1/3}_{H_0^1(D)}, \tag{6.10}$$

where c is a constant, depending only on λ and Λ.

Proof: We define the following diffusion coefficients

$$\frac{1}{a(x)} := A(x) = \begin{cases} 1, & \text{for } 0 < x \le \alpha, \\ 2, & \text{for } \alpha < x < 1, \end{cases}$$

$$\frac{1}{b(x)} := B(x) = \begin{cases} 1, & \text{for } 0 < x \le \beta, \\ 2, & \text{for } \beta < x < 1, \end{cases}$$

where $\alpha, \beta \in (0,1)$, and compute

$$\|A-B\|_{L_2(0,1)} = |\alpha - \beta|^{1/2}. \tag{6.11}$$

Let $g(t) := \frac{1-t^2/2}{2-t}$. Then, a simple calculation gives

$$\gamma_a = g(\alpha), \qquad \gamma_b = g(\beta), \tag{6.12}$$

where γ_a and γ_b are defined by (6.2). We denote by α_0 the point where g achieves its minimum in $(0,1)$. Then, we have

$$g'(\alpha_0) = 1 - 2\alpha_0 + \alpha_0^2/2 = 0 \quad \text{and} \quad \alpha_0 = 2\sqrt{2} - 2. \tag{6.13}$$

We fix α as α_0. Since $g(\alpha_0) = \alpha_0$, we have $\gamma_a = \alpha_0$.

We now bound $\eta := \gamma_a - \gamma_b = \alpha_0 - \gamma_b$ from above. In fact, using (6.11) and (6.13), we have

$$|\eta| = g(\beta) - g(\alpha_0) = \frac{(\alpha_0-\beta)^2}{2(2-\beta)} < \frac{1}{2}(\alpha_0-\beta)^2 = \frac{1}{2}\|A-B\|^4_{L_2(0,1)}. \tag{6.14}$$

Recall that

$$E'(x) = -(A(x)-B(x))(x-\gamma_a) + B(x)(\gamma_a - \gamma_b) = -(A(x)-B(x))(x-\alpha_0) + B(x)\eta. \tag{6.15}$$

Therefore, using (6.11) and (6.14) , we have

$$\begin{aligned}
\|E'\|^2_{L_2(0,1)} &\le 2\int_0^1 (A(x)-B(x))^2(x-\alpha_0)^2\,dx + 2\eta^2 \int_0^1 B^2(x)\,dx \\
&\le 2\left|\int_{\alpha_0}^{\beta} (x-\alpha_0)^2\,dx\right| + 8\eta^2 = \frac{2}{3}|\beta-\alpha_0|^3 + 8\eta^2 = \frac{2}{3}\|A-B\|^6_{L_2(0,1)} + 8\eta^2 \\
&\le \frac{2}{3}\|A-B\|^6_{L_2(0,1)} + 2\|A-B\|^8_{L_2(0,1)} \le C\|A-B\|^6_{L_2(0,1)},
\end{aligned} \tag{6.16}$$

where C depends only on λ, Λ. This completes the proof. □

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