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Generalized Gauss–Radau and Gauss–Lobatto formulas with Jacobi weight functions

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Abstract We compute semi-explicitly or recursively the weights of the generalized Gauss–Radau and Gauss–Lobatto quadratures with Jacobi weight functions.

Keywords Generalized Gauss–Radau quadrature · Generalized Gauss–Lobatto quadrature · Jacobi polynomials

Mathematics Subject Classification 65D32 · 65D30 · 41A55

1 Introduction

In this paper, we consider the quadratures

$$\int_{-1}^{1} \omega(x) f(x)\, dx \approx \sum_{i=0}^{r} e_i f^{(i)}(1) + \sum_{j=1}^{m} d_j f(x_j), \quad r \ge 0, \quad m \ge 1, \tag{1}$$

$$\int_{-1}^{1} \omega(x) f(x)\, dx \approx \sum_{i=0}^{k} c_i f^{(i)}(-1) + \sum_{j=1}^{m} b_j f(y_j), \quad k \ge 0, \quad m \ge 1, \tag{2}$$

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and for $k, r \geq 0$, $m \geq 1$, the quadrature

$$\int_{-1}^{1} \omega(x) f(x)\, dx \approx \sum_{i=0}^{k} \lambda_i f^{(i)}(-1) + \sum_{j=1}^{m} \omega_j f(z_j) + \sum_{i=0}^{r} \mu_i f^{(i)}(1), \tag{3}$$

with weight function ω, that involve not only the values of the integrand f inside the interval $(-1, 1)$, but also the values of the derivatives of f at the endpoints 1 and/or -1. These formulas are often used in applications, for example, in solving boundary value problems by spectral methods, see [10]. We are interested in quadratures (1), (2) and (3) that are exact for all univariate polynomials from

$$\pi_\ell(\mathbb{R}) := \{a_0 + a_1 x + \cdots + a_\ell x^\ell,\ \ a_0, \ldots, a_\ell \in \mathbb{R}\}$$

of degree ℓ as high as possible. Recall that the number $\ell \in \mathbb{N}$ for which a quadrature is exact for all polynomials in $\pi_\ell(\mathbb{R})$, and there is a polynomial from $\pi_{\ell+1}(\mathbb{R})$ for which the formula is not exact, is called *Algebraic Degree of Precision* (ADP) of this quadrature. Clearly, the ADP (1) $\leq 2m + r$ and ADP (2) $\leq 2m + k$, since (1) and (2) are not exact for the polynomials

$$(1-x)^{r+1} \prod_{j=1}^{m} (x - x_j)^2, \quad \text{and} \quad (1+x)^{k+1} \prod_{j=1}^{m} (x - y_j)^2,$$

respectively. Formulas of type (1) and (2) with ADP(1)$= 2m+r$ and ADP(2)$= 2m+k$, respectively, are called *generalized Gauss–Radau* formulas. On the other hand, the ADP (3)$\leq 2m + k + r + 1$, since it does not integrate exactly the polynomial

$$(1-x)^{r+1}(1+x)^{k+1} \prod_{j=1}^{m} (x - z_j)^2.$$

Formula of type (3) with ADP (3)$= 2m+k+r+1$ is called *generalized Gauss–Lobatto* formula.

The generalized Gauss–Radau and Gauss–Lobatto quadratures have been introduced recently in [3], where computational methods have been developed for generating their nodes and weights. The existence and uniqueness of these formulas and the characterization of their internal nodes and weights is well known, see [4]. However, explicit formulas for all of their endpoint weights are known only for few weight functions ω and values of r and/or k. Such formulas have been given in [4] for the case $\omega(x) = 1$, in [6] for the cases

$$\omega(x) = \begin{cases} (1-x^2)^{-1/2}, \\ (1-x^2)^{1/2}, \\ (1-x)^{-1/2}(1+x)^{1/2}, \\ (1-x)^{1/2}(1+x)^{-1/2}, \end{cases}$$

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$k = 1$ and/or $r = 1$, and in [5] for the Jacobi weight function

$$\omega(x) = \omega^{(\alpha,\beta)}(x) := (1-x)^\alpha (1+x)^\beta, \quad \alpha > -1, \beta > -1, \tag{4}$$

and $k = 0$ (which is the classical Gauss–Radau formula). In addition, it has been proven recently, see [8], that in the case of a general weight function ω and any k and/or r all interior and left endpoint weights of the generalized Gauss–Radau and Gauss–Lobatto quadratures are positive, while all right endpoint weights are with alternating signs.

In this paper, we compute semi-explicitly the interior weights and give recursive relations for the endpoint weights of the generalized Gauss–Radau quadratures (1) and (2) and the generalized Gauss–Lobatto formula (3) for the Jacobi weight function (4) for any value of k and/or r. In contrast to [3], where numerical integration has been used to evaluate certain integrals, we obtain exact formulas. Similar approach has been used in [1,9] for calculating the weights of the generalized Gauss–Radau quadrature (1) in the cases of weight function $\omega = \omega^{(0,n/2-1)}$, $n \in \mathbb{N}$, and $r = 0, 1$.

The paper is organized as follows. In Sect. 2, we provide several results that we use for the construction of the generalized Gauss–Radau and Gauss–Lobatto quadrature formulas. The computation of the weights of (1) and (2) is performed in Sect. 3. The generalized Gauss–Lobatto quadrature (3) is presented in Sect. 4.

2 Preliminaries

In this section, we discuss some notation and preliminary results, which we need in the computation of the weights and nodes of the generalized Gauss–Radau formulas (1), (2), and the generalized Gauss–Lobatto quadrature (3).

Let us first recall that the Pochhammer symbol $(a)_j$ is defined by

$$(a)_0 := 1, \quad (a)_j := a(a+1)\cdots(a+j-1) = \frac{\Gamma(a+j)}{\Gamma(a)}, \quad j = 1, 2, \ldots,$$

with Γ being the Gamma function.

It is a well known fact, see [11], that the Jacobi polynomials $P_m^{(\alpha,\beta)}$, $\alpha, \beta > -1$ are orthogonal on $(-1, 1)$ with respect to the weight function $\omega^{(\alpha,\beta)}(x) := (1-x)^\alpha(1+x)^\beta$. We normalize them such that

$$P_m^{(\alpha,\beta)}(1) = \frac{(\alpha+1)_m}{m!}.$$

In what follows, we use the appropriately scaled Jacobi polynomial $R_m^{(\alpha,\beta)}$, that is

$$R_m^{(\alpha,\beta)}(x) := \frac{m!}{(\alpha+1)_m} P_m^{(\alpha,\beta)}(x), \tag{5}$$

and investigate some of its properties. Clearly, $R_m^{(\alpha,\beta)}(1) = 1$, and if we use the symmetry relation

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$$P_m^{(\alpha,\beta)}(-t) = (-1)^m P_m^{(\beta,\alpha)}(t),$$

for Jacobi polynomials, we obtain

$$R_m^{(\alpha,\beta)}(-t) = (-1)^m \frac{(\beta+1)_m}{(\alpha+1)_m} R_m^{(\beta,\alpha)}(t). \tag{6}$$

In particular, we have that

$$R_m^{(\alpha,\beta)}(-1) = (-1)^m \frac{(\beta+1)_m}{(\alpha+1)_m}. \tag{7}$$

We will need the values of the derivatives of $R_m^{(\alpha,\beta)}$ at 1 and -1. We compute them using the fact that $R_m^{(\alpha,\beta)}(1) = 1$, relation (7), and the formula

$$\frac{d^\ell}{dx^\ell} P_m^{(\alpha,\beta)} = \frac{\Gamma(\alpha+\beta+m+\ell+1)}{2^\ell \Gamma(\alpha+\beta+m+1)} P_{m-\ell}^{(\alpha+\ell,\beta+\ell)}.$$

If we denote by $\Omega_m(\alpha,\beta,\ell)$ the function

$$\Omega_m(\alpha,\beta,\ell) = \begin{cases} (\alpha+1)_m, & \ell = 0, \\ \frac{1}{2^\ell}(m-\ell+1)_\ell(\alpha+\beta+m+1)_\ell(\alpha+\ell+1)_{m-\ell}, & 0 < \ell \le m, \\ 0, & \ell > m, \end{cases} \tag{8}$$

and by $\Psi_m(\alpha,\beta,\ell)$ the function

$$\Psi_m(\alpha,\beta,\ell) = \begin{cases} (\beta+1)_m, & \ell = 0, \\ \frac{1}{2^\ell}(-1)^\ell(m-\ell+1)_\ell(\alpha+\beta+m+1)_\ell(\beta+\ell+1)_{m-\ell}, & 0 < \ell \le m, \\ 0, & \ell > m, \end{cases} \tag{9}$$

we derive that

$$\frac{d^\ell}{dx^\ell} R_m^{(\alpha,\beta)}(1) = \frac{\Omega_m(\alpha,\beta,\ell)}{(\alpha+1)_m}, \tag{10}$$

and

$$\frac{d^\ell}{dx^\ell} R_m^{(\alpha,\beta)}(-1) = (-1)^m \frac{\Psi_m(\alpha,\beta,\ell)}{(\alpha+1)_m}. \tag{11}$$

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We also will need the derivatives of the polynomials

$$(1-x)^p R_m^{(\alpha,\beta)}(x), \quad (1+x)^p R_m^{(\alpha,\beta)}(x),$$

and their values at -1 and 1, respectively. The computation of the first one utilizes the product rule and Eq. (11), which give

$$\frac{d^\ell}{dx^\ell}\left((1-\cdot)^p R_m^{(\alpha,\beta)}(\cdot)\right)(-1) = \frac{(-1)^m}{(\alpha+1)_m}\sum_{j=0}^{\ell}\binom{\ell}{j}2^{p-j}(-1)^j(p-j+1)_j \Psi_m(\alpha,\beta,\ell-j).$$

Now, if we denote by $\Lambda_m(\alpha,\beta,\ell,p)$ the function

$$\Lambda_m(\alpha,\beta,\ell,p) = \begin{cases} 2^p\Psi_m(\alpha,\beta,0), & \ell=0, \\ \sum_{j=0}^{\ell}\binom{\ell}{j}2^{p-j}(-1)^j(p-j+1)_j\Psi_m(\alpha,\beta,\ell-j), & \ell>0, \end{cases} \tag{12}$$

we obtain the following formula for the value of the derivative at -1,

$$\frac{d^\ell}{dx^\ell}\left((1-\cdot)^p R_m^{(\alpha,\beta)}(\cdot)\right)(-1) = \frac{(-1)^m}{(\alpha+1)_m}\Lambda_m(\alpha,\beta,\ell,p). \tag{13}$$

Likewise, using the product rule and relation (10), we derive the formula

$$\frac{d^\ell}{dx^\ell}\left((1+\cdot)^p R_m^{(\alpha,\beta)}(\cdot)\right)(1) = \frac{1}{(\alpha+1)_m}\Theta_m(\alpha,\beta,\ell,p),$$

where Θ_m is defined as

$$\Theta_m(\alpha,\beta,\ell,p) := \begin{cases} 2^p\Omega_m(\alpha,\beta,0), & \ell=0, \\ \sum_{j=0}^{\ell}\binom{\ell}{j}2^{p-j}(p-j+1)_j\Omega_m(\alpha,\beta,\ell-j), & \ell>0. \end{cases} \tag{14}$$

Next, in the derivation of the coefficients of the generalized Gauss–Radau and the generalized Gauss–Lobatto quadratures, we need an explicit formula for the values of certain integrals of $R_m^{(\alpha,\beta)}$, given by the following lemma.

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Lemma 1 *If $R_m^{(\alpha,\beta)}$ is the polynomial defined in (5), then we have*

$$I_m^{(\alpha,\beta)}(\delta) := \int_{-1}^{1}(1-x)^\delta(1+x)^\beta R_m^{(\alpha,\beta)}(x)\,dx$$
$$= 2^{\delta+\beta+1}\frac{\Gamma(\delta+1)\Gamma(\beta+1+m)\Gamma(\alpha-\delta+m)}{(\alpha+1)_m\Gamma(\beta+\delta+2+m)\Gamma(\alpha-\delta)}, \tag{15}$$
$$J_m^{(\alpha,\beta)}(\delta) := \int_{-1}^{1}(1-x)^\alpha(1+x)^\delta R_m^{(\alpha,\beta)}(x)\,dx$$
$$= (-1)^m\frac{(\beta+1)_m}{(\alpha+1)_m}I_m^{(\beta,\alpha)}(\delta). \tag{16}$$

Proof Formula (15) is known, see equation 7.391.4 from [7], or relation (2) in [2], page 284. The formula for $J_m^{(\alpha,\beta)}$ is also known, see equation 7.391.3 in [7]. However, here we have decided to use a simple change of variables to connect $I_m^{(\alpha,\beta)}$ with the value of $J_m^{(\alpha,\beta)}$. We set $x=-t$ and apply the symmetry relation (6) to derive

$$J_m^{(\alpha,\beta)}(\delta) = \int_{-1}^{1}(1-t)^\delta(1+t)^\alpha R_m^{(\alpha,\beta)}(-t)\,dt$$
$$= (-1)^m\frac{(\beta+1)_m}{(\alpha+1)_m}\int_{-1}^{1}(1-t)^\delta(1+t)^\alpha R_m^{(\beta,\alpha)}(t)\,dt = (-1)^m\frac{(\beta+1)_m}{(\alpha+1)_m}I^{(\beta,\alpha)}(\delta).$$

We conclude this section by discussing the Gaussian quadrature

$$\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta f(x)\,dx = \sum_{j=1}^{m}a_j^{(\alpha,\beta)}f(x_j^{(\alpha,\beta)})$$

for the interval $(-1,1)$ with Jacobi weight $(1-x)^\alpha(1+x)^\beta$. Recall that this quadrature has nodes $x_j^{(\alpha,\beta)}$, $j=1,\dots,m$, the zeroes of the Jacobi polynomials $P_m^{(\alpha,\beta)}$, and coefficients $a_j^{(\alpha,\beta)}$, given by the formula, see [11, p. 352, Eq. (15.3.1)],

$$a_j^{(\alpha,\beta)} = 2^{\alpha+\beta+1}\frac{\Gamma(m+\alpha+1)\Gamma(m+\beta+1)}{\Gamma(m+1)\Gamma(m+\alpha+\beta+1)}\left(1-(x_j^{(\alpha,\beta)})^2\right)^{-1}$$
$$\times\left\{\frac{d}{dx}P_m^{(\alpha,\beta)}(x_j^{(\alpha,\beta)})\right\}^{-2}.$$

We use another expression for the coefficients, which we obtain by applying the relation (4.5.7) from [11],

$$\frac{d}{dx}P_m^{(\alpha,\beta)}(x_j^{(\alpha,\beta)}) = -2\frac{(m+1)(m+\alpha+\beta+1)}{2m+\alpha+\beta+2}\cdot\frac{P_{m+1}^{(\alpha,\beta)}(x_j^{(\alpha,\beta)})}{1-(x_j^{(\alpha,\beta)})^2}.$$

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The formula we use is

$$a_j^{(\alpha,\beta)} = \Delta_m(\alpha,\beta) \cdot \frac{1-(x_j^{(\alpha,\beta)})^2}{\left[P_{m+1}^{(\alpha,\beta)}(x_j^{(\alpha,\beta)})\right]^2}, \tag{17}$$

where Δ_m is given by

$$\Delta_m(\alpha,\beta) := 2^{\alpha+\beta-1}\frac{\Gamma(m+\alpha+1)\Gamma(m+\beta+1)(2m+\alpha+\beta+2)^2}{\Gamma(m+1)\Gamma(m+\alpha+\beta+1)(m+1)^2(m+\alpha+\beta+1)^2}. \tag{18}$$

3 Generalized Gauss–Radau formulae

In this section, we provide semi-explicit or recursive formulas for the weights of the generalized Gauss–Radau quadratures (1) and (2) with the Jacobi weight (4). The following theorem holds.

Theorem 1 *There is a unique quadrature formula of type* (1) *with ADP*$= 2m + r$. *Its nodes* $\{x_j\}$, $j = 1,\ldots,m$, *are the zeroes of the Jacobi polynomial* $P_m^{(\alpha+r+1,\beta)}$. *Its weights* e_i, $i = 0,\ldots r$, *are given by the recursive relation*

$$e_r = \frac{(-1)^r}{r!} I_m^{(\alpha+r+1,\beta)}(\alpha+r),$$
$$e_i = \frac{(-1)^i}{i!} I_m^{(\alpha+r+1,\beta)}(\alpha+i) - \frac{1}{(\alpha+r+2)_m}\sum_{s=i+1}^{r} e_s\binom{s}{i}\Omega_m(\alpha+r+1,\beta,s-i),$$

$i = r-1,\ldots,0$, *where* Ω_m *is defined in* (8) *and* $I_m^{(\alpha,\beta)}$ *is computed in Lemma 1. The coefficients* d_j, $j = 1,\ldots,m$, *are*

$$d_j = \Delta_m(\alpha+r+1,\beta)\cdot\frac{(1+x_j)}{(1-x_j)^r\left[P_{m+1}^{(\alpha+r+1,\beta)}(x_j)\right]^2},$$

with Δ_m *defined in* (18).

Proof We do not discuss here the existence and uniqueness of the generalized Gauss–Radau formula (1), since this is a classical result, but rather focus our effort on the computation of its weights. First, we calculate the coefficients d_j. We apply (1) to $(1-x)^{r+1}S(x)$, where S is any polynomial from $\pi_{2m-1}(\mathbb{R})$. It follows that

$$\int_{-1}^{1}(1-x)^{\alpha+r+1}(1+x)^{\beta}S(x)\,dx = \sum_{j=1}^{m} d_j(1-x_j)^{r+1}S(x_j),$$

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and therefore this is the unique Gaussian quadrature on $(-1, 1)$ with weight function $\omega(x) = (1-x)^{\alpha+r+1}(1+x)^{\beta}$. The nodes of this formula x_j are the zeroes of $P_m^{(\alpha+r+1,\beta)}$ and the coefficients are given by (17). Hence, the nodes of (1) are the zeroes of $P_m^{(\alpha+r+1,\beta)}$, and for $j = 1, \dots, m$, the interior weights d_j are computed as

$$d_j = a_j^{(\alpha+r+1,\beta)}(1-x_j)^{-r-1} = \Delta_m(\alpha+r+1,\beta)\cdot\frac{(1+x_j)}{(1-x_j)^r\left[P_{m+1}^{(\alpha+r+1,\beta)}(x_j)\right]^2}.$$

Next, we compute the weights e_i, $i = 0, \dots, r$. We start with e_r. We apply (1) to the polynomial $(1-x)^r R_m^{(\alpha+r+1,\beta)}(x) \in \pi_{m+r}(\mathbb{R})$ and obtain, in view of Eqs. (15)–(16),

$$e_r = \frac{(-1)^r}{r!} I_m^{(\alpha+r+1,\beta)}(\alpha+r).$$

We calculate the rest of the weights e_i, $0 \le i < r$, recursively. Let us assume that we have already computed e_j, $i < j \le r$. Then, to compute e_i, we apply the quadrature (1) to the polynomial $(1-x)^i R_m^{(\alpha+r+1,\beta)}(x)$ and derive the equation

$$(-1)^i i! \sum_{s=i}^{r} e_s \binom{s}{i} \frac{d^{s-i}}{dx^{s-i}} R_m^{(\alpha+r+1,\beta)}(1) = I_m^{(\alpha+r+1,\beta)}(\alpha+i).$$

It follows from (10) that

$$\frac{1}{(\alpha+r+2)_m}\sum_{s=i}^{r} e_s \binom{s}{i} \Omega_m(\alpha+r+1,\beta,s-i) = \frac{(-1)^i}{i!} I_m^{(\alpha+r+1,\beta)}(\alpha+i),$$

and therefore

$$e_i = \frac{(-1)^i}{i!} I_m^{(\alpha+r+1,\beta)}(\alpha+i) - \frac{1}{(\alpha+r+2)_m}\sum_{s=i+1}^{r} e_s \binom{s}{i} \Omega_m(\alpha+r+1,\beta,s-i),$$

$i = r-1, \dots, 0$.

We conclude the discussion of the generalized Gauss–Radau formula (1) by performing several numerical experiments, recorded in Tables 1, 2, 3, 4, 5, 6, 7 and 8, where we list the weights and nodes of quadrature (1) for various values of r, m, α and β. Note that our experiments agree with the theoretical results in [8], where it has been proven that for a general weight function ω, $d_j > 0$, $j = 1, \dots, m$, and $(-1)^i e_i > 0$, $i = 0, \dots, r$.

The code for all numerical experiments here and later in the paper is written in Matlab. In particular, the implementation of the Jacobi polynomials uses the Matlab built-in function *jacobiP*. The computation of the zeros of the Jacobi polynomials is performed using a third-party package written by John Burkardt (see https://people.sc.fsu.edu/jburkardt/msrc/jacobipolynomial/jacobipolynomial.html). While certain optimization

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Table 1 Formula (1), $m = 7, r = 3, \alpha = \beta = \frac{1}{2}$

j	1	2	3	4	5	6	7
d_j	0.03238	0.11691	0.22065	0.30296	0.33112	0.29309	0.20279
x_j	−0.94848	−0.79896	−0.56602	−0.27237	0.05350	0.38030	0.67835

Table 2 Formula (1), $m = 7, r = 3, \alpha = \beta = \frac{1}{2}$

i	0	1	2	3
e_i	0.0708992	−0.0061166	0.0002311	−0.0000035

Table 3 Formula (1), $m = 7, r = 4, \alpha = 1, \beta = \frac{1}{2}$

j	1	2	3	4	5	6	7
d_j	0.03822	0.13497	0.24515	0.31812	0.32161	0.25601	0.15249
x_j	−0.95405	−0.82006	−0.60931	−0.33946	−0.03294	0.28537	0.59241

Table 4 Formula (1), $m = 7, r = 4, \alpha = 1, \beta = \frac{1}{2}$

i	0	1	2	3	4
e_i	0.0419195	−0.0057068	0.0003572	−0.0000116	0.0000002

Table 5 Formula (1), $m = 7, r = 4, \alpha = -\frac{1}{3}, \beta = \frac{9}{5}$

j	1	2	3	4	5	6	7
d_j	0.00208	0.01930	0.07713	0.20191	0.40414	0.66884	0.98217
x_j	−0.89314	−0.71165	−0.46606	−0.17707	0.13098	0.43249	0.70400

Table 6 Formula (1), $m = 7, r = 4, \alpha = -\frac{1}{3}, \beta = \frac{9}{5}$

i	0	1	2	3	4
e_i	1.5610406	−0.0971652	0.0039814	−0.0000937	0.0000010

steps could have been implemented for the overall improvement of the precision in our experiments, we have found these to be unnecessary for the results presented below. Of course, as the degree of the polynomials increases, one has to ensure the proper computation of the functions involving sums and ratios of Pochhammer and Gamma functions as they may introduce a mixture of very large and very small terms.

Finally, in Table 9, we demonstrate the ADP of formula (1) with parameters $\alpha = \frac{3}{4}$, $\beta = \frac{1}{3}$. We use (1) to compute the corresponding integral for the function $f(x) = (1+x)^{2m+r}$ and obtain that the quadrature is exact for that chosen f up to the machine

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Table 7 Formula (1), $m = 7, r = 4, \alpha = \frac{4}{3}, \beta = -\frac{1}{5}$

j	1	2	3	4	5	6	7
d_j	0.33698	0.53228	0.57707	0.51045	0.37676	0.22609	0.10099
x_j	−0.97863	−0.86955	−0.67549	−0.41316	−0.10504	0.22314	0.54721

Table 8 Formula (1), $m = 7, r = 4, \alpha = \frac{4}{3}, \beta = -\frac{1}{5}$

i	0	1	2	3	4
e_i	0.0188874	−0.0029486	0.0002055	−0.0000073	0.0000001

Table 9 Errors for (1) with $\alpha = \frac{3}{4}, \beta = \frac{1}{3}$ for $f(x) = (1+x)^{2m+r}$

m	1	2	3	4	5	6	7
$r = 2$	$7.1e^{-15}$	$7.1e^{-15}$	$2.8e^{-14}$	0	$6.5e^{-13}$	$3.3e^{-12}$	$7.3e^{-12}$
$r = 4$	$1.4e^{-14}$	$1.1e^{-14}$	$9.2e^{-14}$	$9.1e^{-13}$	$2.3e^{-12}$	$2.5e^{-12}$	$2.8e^{-11}$

error. In this computation, as well as in the computations for Tables 18 and 27, we evaluate the integral $\int_{-1}^{1}(1-x)^p(1+x)^q\,dx$ exactly for various values of p and q using the formula

$$\int_{-1}^{1}(1-x)^p(1+x)^q\,dx = 2^{p+q+1}\frac{\Gamma(q+1)\Gamma(p+1)}{\Gamma(p+q+2)},$$

where Γ is the Gamma function.

The next theorem provides semi-explicit or recursive formulas for the weights of the generalized Gauss–Radau quadrature (2). Since the derivations are similar to the ones in Theorem 1, we omit the proof.

Theorem 2 *There is a unique quadrature formula of type* (2) *with ADP= $2m+k$. Its nodes $\{y_j\}$, $j = 1,\ldots,m$, are the zeroes of the Jacobi polynomial $P_m^{(\alpha,\beta+k+1)}$. Its weights c_i, $i = 0,\ldots k$, are given by the recursive relation*

$$c_k = \frac{1}{k!}I_m^{(\beta+k+1,\alpha)}(\beta+k),$$

and for $i = k-1,\ldots,0$,

$$c_i = \frac{1}{i!}I_m^{(\beta+k+1,\alpha)}(\beta+i) - \frac{1}{(\beta+k+2)_m}\sum_{s=i+1}^{k} c_s\binom{s}{i}\Psi_m(\alpha,\beta+k+1,s-i),$$

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where Ψ_m is defined in (9) and the integral $I_m^{(\alpha,\beta)}$ is computed in Lemma 1. The coefficients b_j, $j = 1, \ldots, m$, are

$$b_j = \Delta_m(\alpha, \beta + k + 1) \cdot \frac{(1 - y_j)}{(1 + y_j)^k \left[P_{m+1}^{(\alpha,\beta+k+1)}(y_j) \right]^2},$$

where Δ_m is defined in (18).

Here, we also provide several tables, see Tables 10, 11, 12, 13, 14, 15, 16 and 17, with computed weights and nodes of quadrature (2) for various values of k, m, α and β. Again, the experiments confirm the theoretical results in [8], where it has been proven that for a general weight function ω, $b_j > 0$, $j = 1, \ldots, m$, and $c_i > 0$, $i = 0, \ldots, k$.

Table 10 Formula (2), $m = 7$, $k = 3$, $\alpha = \beta = \frac{1}{2}$

j	1	2	3	4	5	6	7
b_j	0.20279	0.29309	0.33112	0.30296	0.22065	0.11691	0.03238
y_j	−0.67835	−0.38030	−0.05350	0.27237	0.56602	0.79896	0.94848

Table 11 Formula (2), $m = 7$, $k = 3$, $\alpha = \beta = \frac{1}{2}$

i	0	1	2	3
c_i	0.0708992	0.0061166	0.0002311	0.0000035

Table 12 Formula (2), $m = 7$, $k = 4$, $\alpha = 1$, $\beta = \frac{1}{2}$

j	1	2	3	4	5	6	7
b_j	0.27561	0.33369	0.31886	0.24243	0.14130	0.05616	0.01045
y_j	−0.63645	−0.34258	−0.03015	0.27757	0.55538	0.78008	0.93274

Table 13 Formula (2), $m = 7$, $k = 4$, $\alpha = 1$, $\beta = \frac{1}{2}$

i	0	1	2	3	4
c_i	0.1300080	0.0145270	0.0007959	0.0000232	0.0000003

Table 14 Formula (2), $m = 7$, $k = 4$, $\alpha = 1$, $\beta = -\frac{1}{2}$

j	1	2	3	4	5	6	7
b_j	0.80759	0.55787	0.36880	0.21600	0.10451	0.03656	0.00630
y_j	−0.69267	−0.40612	−0.08937	0.23025	0.52353	0.76335	0.92744

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Table 15 Formula (2), $m = 7, k = 4, \alpha = 1, \beta = -\frac{1}{2}$

i	0	1	2	3	4
c_i	1.6736076	0.0915782	0.0037751	0.0000912	0.0000010

Table 16 Formula (2), $m = 7, k = 4, \alpha = \frac{3}{5}, \beta = -\frac{1}{8}$

j	1	2	3	4	5	6	7
b_j	0.43355	0.40061	0.33520	0.24504	0.14941	0.06818	0.01644
y_j	−0.65943	−0.36081	−0.03789	0.28183	0.56921	0.79768	0.94600

Table 17 Formula (2), $m = 7, k = 4, \alpha = \frac{3}{5}, \beta = -\frac{1}{8}$

i	0	1	2	3	4
c_i	0.4233420	0.0355566	0.0017249	0.0000466	0.0000006

Table 18 Errors for (2) with $\alpha = \frac{2}{3}, \beta = \frac{4}{5}$ for $f(x) = (1-x)^{2m+k}$

m	1	2	3	4	5	6	7
$k = 2$	$4.4e^{-16}$	0	$7.1e^{-15}$	$7.1e^{-14}$	$5.7e^{-14}$	$1.1e^{-12}$	$4.1e^{-12}$
$k = 4$	$1.2e^{-14}$	$3.6e^{-14}$	$3.1e^{-13}$	$2.8e^{-14}$	$3.2e^{-12}$	$1.1e^{-11}$	$4.9e^{-11}$

In the end, we demonstrate the ADP of formula (2) with parameters $\alpha = \frac{2}{3}, \beta = \frac{4}{5}$ by using this quadrature to compute the corresponding integral of the function $f(x) = (1-x)^{2m+k}$. The results are recorded in Table 18.

4 Generalized Gauss–Lobatto formula

In this section, we provide semi-explicit or recursive formulas for the weights of the generalized Gauss–Lobatto quadrature (3) with the Jacobi weight (4).

Theorem 3 *There is a unique quadrature formula of type* (3) *with ADP*$= 2m + k + r + 1$. *Its nodes* $\{z_j\}$, $j = 1, \ldots, m$, *are the zeroes of the Jacobi polynomial* $P_m^{(\alpha+r+1,\beta+k+1)}$. *Its weights* λ_i, $i = 0, \ldots k$, *are given by the recursive relation*

$$\lambda_k = \frac{1}{2^{r+1}k!} I_m^{(\beta+k+1,\alpha+r+1)}(\beta + k),$$

and for $i = k-1, \ldots, 0$,

$$\lambda_i = \frac{I_m^{(\beta+k+1,\alpha+r+1)}(\beta + i)}{2^{r+1}i!} - \frac{2^{-r-1}}{(\beta+k+2)_m} \sum_{s=i+1}^{k} \lambda_s \binom{s}{i} \Lambda_m(\alpha + r + 1, \beta + k + 1, s - i, r + 1),$$

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where Λ_m is defined in (12). Its weights μ_i, $i = 0, \ldots r$, are given by the recursive relation

$$\mu_r = \frac{(-1)^r}{2^{k+1} r!} I_m^{(\alpha+r+1,\beta+k+1)}(\alpha + r),$$

and for $i = r - 1, \ldots, 0$,

$$\mu_i = \frac{(-1)^i I_m^{(\alpha+r+1,\beta+k+1)}(\alpha + i)}{2^{k+1} i!} - \frac{2^{-k-1}}{(\alpha + r + 2)_m} \sum_{s=i+1}^{r} \mu_s \binom{s}{i} \Theta_m(\alpha + r + 1, \beta + k + 1, s - i, k + 1),$$

with Θ_m defined in (14). The interior weights ω_i, $i = 1, \ldots, m$, are

$$\omega_j = \frac{\Delta_m(\alpha + r + 1, \beta + k + 1)}{(1 + z_j)^k (1 - z_j)^r \left[P_{m+1}^{(\alpha+r+1,\beta+k+1)}(z_j)\right]^2}, \quad j = 1, \ldots, m,$$

where Δ_m is defined in (18).

Proof The proof is similar to the proof of Theorem 1, so we provide here the details for the computations of λ_k only. We apply the generalized Gauss–Lobatto quadrature (3) to the polynomial $(1 + x)^k (1 - x)^{r+1} R_m^{(\alpha+r+1,\beta+k+1)}(x)$ and obtain

$$k! \lambda_k 2^{r+1} R_m^{(\alpha+r+1,\beta+k+1)}(-1) = J_m^{(\alpha+r+1,\beta+k+1)}(\beta + k),$$

which together with (7) and Lemma 1 results in

$$\lambda_k = \frac{1}{2^{r+1} k!} I_m^{(\beta+k+1,\alpha+r+1)}(\beta + k).$$

The rest of the weights λ_i, $0 \le i < k$, are computed recursively after the application of (3) to the polynomial $(1 + x)^i (1 - x)^{r+1} R_m^{(\alpha+r+1,\beta+k+1)}(x)$ and the use of the product rule and relation (13). The formula is

$$\frac{(-1)^m}{(\alpha + r + 2)_m} i! \sum_{s=i}^{k} \lambda_s \binom{s}{i} \Lambda_m(\alpha + r + 1, \beta + k + 1, s - i, r + 1) = J_m^{(\alpha+r+1,\beta+k+1)}(\beta + i),$$

where the function Λ_m is defined in (12). Simple computations and the application of (16), lead to the final result for the weight λ_i, $i = k - 1, \ldots, 0$,

$$\lambda_i = \frac{I_m^{(\beta+k+1,\alpha+r+1)}(\beta + i)}{2^{r+1} i!} - \frac{2^{-r-1}}{(\beta + k + 2)_m} \sum_{s=i+1}^{k} \lambda_s \binom{s}{i} \Lambda_m(\alpha + r + 1, \beta + k + 1, s - i, r + 1).$$

Finally, we demonstrate our results by performing several numerical experiments and computing the weights and the nodes of the generalized Gauss–Lobatto formula (3) for particular values of the parameters α, β, k, r and m, see Tables 19, 20, 21, 22, 23, 24, 25 and 26. Note that, as has been proven in [8] for a general weight function

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Table 19 Formula (3), $m = 7, k = 3, r = 2, \alpha = \beta = \frac{1}{2}$

j	1	2	3	4	5	6	7
ω_j	0.14533	0.22408	0.27766	0.29015	0.25809	0.19107	0.10899
z_j	−0.74843	−0.50795	−0.23351	0.05634	0.34064	0.59870	0.81240

Table 20 Formula (3), $m = 7, k = 3, r = 2, \alpha = \beta = \frac{1}{2}$

i	0	1	2	3
λ_i	0.0397785	0.0027084	0.0000843	0.0000011
μ_i	0.0267346	−0.0010539	0.0000137	–

Table 21 Formula (3), $m = 7, k = 4, r = 3, \alpha = 1, \beta = \frac{1}{2}$

j	1	2	3	4	5	6	7
ω_j	0.19237	0.25868	0.28584	0.26654	0.20948	0.13441	0.06430
z_j	−0.72786	−0.49894	−0.24376	0.02466	0.29094	0.53976	0.75807

Table 22 Formula (3), $m = 7, k = 4, r = 3, \alpha = 1, \beta = \frac{1}{2}$

i	0	1	2	3	4
λ_i	0.0605581	0.0049851	0.0002131	0.0000050	0.0000001
μ_i	0.0126249	−0.0008629	0.0000241	−0.0000003	–

Table 23 Formula (3), $m = 7, k = 4, r = 2, \alpha = 1, \beta = -\frac{1}{2}$

j	1	2	3	4	5	6	7
ω_j	0.75564	0.55822	0.40677	0.27474	0.16531	0.08349	0.03102
z_j	−0.75721	−0.52420	−0.25691	0.02739	0.30906	0.56858	0.78884

Table 24 Formula (3), $m = 7, k = 4, r = 2, \alpha = 1, \beta = -\frac{1}{2}$

i	0	1	2	3	4
λ_i	1.3170027	0.0527961	0.0017334	0.0000347	0.0000003
μ_i	0.0040298	−0.0001883	0.0000027	–	–

ω, the weights $\omega_j > 0$, $j = 1, \ldots, m$, $\lambda_i > 0$, $i = 0, \ldots, k$, and $(-1)^\ell \mu_\ell > 0$, $\ell = 0, \ldots, r$.

We also demonstrate the ADP of formula (3) with parameters $\alpha = \frac{2}{5}$, $\beta = \frac{6}{7}$. We apply the latter quadrature to compute the integral for $f(x) = (1+x)^{m+r}(1-x)^{m+k+1}$ and present the errors of the formula in Table 27. □

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Table 25 Formula (3), $m = 7, k = 3, r = 4, \alpha = -\frac{2}{3}, \beta = \frac{7}{8}$

j	1	2	3	4	5	6	7
ω_j	0.04449	0.10100	0.18340	0.29276	0.43139	0.61034	0.89418
z_j	−0.74510	−0.51299	−0.24990	0.02855	0.30436	0.55963	0.77882

Table 26 Formula (3), $m = 7, k = 3, r = 4, \alpha = -\frac{2}{3}, \beta = \frac{7}{8}$

i	0	1	2	3	4
λ_i	0.0061709	0.0004612	0.0000152	0.0000002	–
μ_i	2.7659422	−0.0819212	0.0023042	−0.0000388	0.0000003

Table 27 Errors of (3) with parameters $\alpha = \frac{2}{5}, \beta = \frac{6}{7}$ for the function $f(x) = (1+x)^{m+r}(1-x)^{m+k+1}$

m	1	2	3	4	5	6	7
$r = 2, k = 3$	$8.9e^{-16}$	0	$1.1e^{-16}$	$4.4e^{-16}$	$1.2e^{-15}$	$7.1e^{-15}$	$1.9e^{-15}$
$r = 3, k = 4$	$3.0e^{-15}$	$2.4e^{-15}$	$5.3e^{-15}$	$1.3e^{-15}$	$7.9e^{-15}$	$5.0e^{-15}$	$2.6e^{-15}$
m	8	9	10	11	12	13	14
$r = 2, k = 3$	$2.8e^{-15}$	$5.2e^{-15}$	$7.8e^{-16}$	$5.2e^{-15}$	$7.4e^{-15}$	$7.9e^{-15}$	$1.1e^{-14}$
$r = 3, k = 4$	$9.9e^{-15}$	$2.2e^{-16}$	$1.2e^{-14}$	$4.9e^{-15}$	$6.9e^{-15}$	$1.2e^{-14}$	$1.1e^{-15}$
m	15	16	17	18	19	20	21
$r = 2, k = 3$	$3.1e^{-15}$	$8.0e^{-15}$	$8.3e^{-16}$	$6.8e^{-15}$	$4.8e^{-15}$	$4.1e^{-15}$	$2.2e^{-15}$
$r = 3, k = 4$	$8.5e^{-15}$	$3.6e^{-15}$	$8.2e^{-15}$	$5.7e^{-15}$	$2.3e^{-15}$	$4.4e^{-15}$	$1.1e^{-16}$

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