Quadrature Formula for Computed Tomography

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August 13, 2009

Abstract

We give a bivariate analog of the Micchelli-Rivlin quadrature for computing the integral of a function over the unit disk using its Radon projections.
AMS subject classification: 65D32, 65D30, 41A55
Key Words: Numerical integration, orthogonal polynomials, Gaussian quadratures, Radon projections.

1 Introduction

There are several known classical methods in Computed Tomography for reconstructing a function f from its x-ray or Radon transforms, such as the Fourier reconstruction algorithm, the filtered backprojection algorithm and the so called algebraic methods. The latter give an approximation $\tilde{f} = \sum_{i=1}^{N} c_i\psi_i \in span\{\psi_1, \ldots, \psi_N\}$ to f with the same x-ray transforms as those of f. One of the main advantages of the algebraic methods is the freedom in the choice of the basis functions $\{\psi_i\}$.

Recently, a new algebraic method was presented in [7, 8, 9], where the functions $\{\psi_i\}$ are selected to be polynomials in $\mathbb{R}^n$. The method is based on the expansion of f, defined on a ball in $\mathbb{R}^n$, using orthonormal polynomials, where the coefficients of this expansion can be represented as integrals involving the Radon transform of f. The algorithm gives an approximation to f by truncating the series expansion and discretizing the coefficients $\{c_i\}$, and thus the performance of the method heavily relies on a good approximation of these coefficients. In this paper, we present a quadrature formula for computing the c_i's that is exact for all bivariate polynomials of degree as high as possible.

2 Preliminaries

We consider the space $L_2(B)$ of bivariate square integrable functions defined on the unit disk $B := \{(x, y) : \ x^2 + y^2 \leq 1\}$. It is a well known fact (see [1] or [5]) that the set of

*The author was supported by the Sofia University Research grant # 135/2008 and by Swiss-NSF Scopes Project IB7320-111079.

†This work has been supported in part by the Bulgarian Science Fund Grant VU-I-303/2007, the NSF Grant #DMS-0810869, and by Award # KUS-C1-016-04, made by King Abdullah University of Science and Technology (KAUST).

polynomials $\{U_{k,n}\}_{n=0,k=0}^{\infty,\ \ n}$, defined by

$$U_{k,n}(x,y) := \frac{1}{\sqrt{\pi}} U_n\left(x\cos(\theta_{k,n}) + y\sin(\theta_{k,n})\right), \quad \theta_{k,n} := \frac{k\pi}{n+1},$$

where

$$U_n(\cos\theta) := \frac{\sin(n+1)\theta}{\sin\theta},$$

is the Chebyshev polynomial of second kind, form a complete orthonormal system for $L_2(B)$. It can be shown that the coefficients $c_{k,n}(f)$ in the expansion of f,

$$f = \sum_{n=0}^{\infty}\sum_{k=0}^{n} c_{k,n}(f) U_n\left(x\cos(\theta_{k,n}) + y\sin(\theta_{k,n})\right), \tag{2.1}$$

with respect to this system are

$$c_{k,n}(f) := \frac{1}{\pi}\int_B f(x,y)U_{k,n}(x,y)\,dxdy = \frac{1}{\pi}\int_{-1}^{1} \mathcal{R}(f;t,\theta_{k,n})U_n(t)\,dt, \tag{2.2}$$

where $\mathcal{R}(f;t,\theta)$ is the Radon transform of f. Recall that the *Radon transform* $\mathcal{R}(f;t,\theta)$, $t \in (-1,1)$, $\theta \in [0,\pi)$ of a function f, defined on the unit ball B, is given by the integral of f along the line segment $I := I(t,\theta) = \{(x,y):\ x\cos\theta + y\sin\theta = t\} \cap B$, namely,

$$\mathcal{R}(f;t,\theta) := \int_{I(t,\theta)} f(x,y)\,ds = \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} f(t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta)\,ds.$$

In view of formula (2.2), the problem of optimal recovery of f is equivalent to the problem of a selection of quadrature formula for the second integral in (2.2) which is exact for polynomials of degree as high as possible. The existing algorithms utilize the fact, see [2], that for every polynomial Q in two variables of degree N, its Radon transform can be represented as $\mathcal{R}(Q;t,\theta) = \sqrt{1-t^2}Q_\theta(t)$, where Q_θ is a polynomial in one variable of the same degree N whose coefficients are trigonometric polynomials in θ. Then, for every polynomial Q, formula (2.2) becomes

$$\begin{aligned} c_{k,n}(Q) &= \frac{1}{\pi}\int_{-1}^{1}\sqrt{1-t^2}U_n(t)Q_{\theta_{k,n}}(t)\,dt \approx \sum_{j=1}^{m} a_j U_n(\eta_j)Q_{\theta_{k,n}}(\eta_j) \\ &= \sum_{j=1}^{m} \frac{a_j U_n(\eta_j)}{\sqrt{1-\eta_j^2}}\mathcal{R}(Q;\eta_j,\theta_{k,n}), \end{aligned} \tag{2.3}$$

where the discretization of $\{c_{k,n}\}$ is done using the Gaussian formula with m nodes for the interval $[-1,1]$ with weight $\mu(t) = \sqrt{1-t^2}$. However, this formula is not accurate enough, since it is exact for polynomials of degree $2m-1$, and therefore for polynomials Q of degree only $2m-n-1$. In particular, when using $m = n+1$ Radon projections, we obtain the formula, currently used in the existing algorithms in [9], that is exact for polynomials Q of degree only $n+1$. The main result of this paper is the derivation of a formula for numerical integration of the Fourier-Chebyshev coefficients $c_{k,n}(f)$ that is exact for polynomials of highest possible degree. We obtain a quadrature, see (4.5), that uses $n+1$ Radon projections and is exact for all bivariate polynomials of degree $3n+1$. This formula can be viewed as a two-dimensional analog to the Micchelli-Rivlin quadrature from [6].

3 The one dimensional case

Relation (2.3) shows that the discretization of $c_{k,n}(f)$ is closely related to the investigation of quadratures of the form

$$\int_a^b \mu(t)P_n(t)g(t)\,dt \approx \sum_{j=1}^{m} a_j g(x_j), \quad a < x_1 < \cdots < x_m < b, \tag{3.1}$$

where P_n is a polynomial of degree n. We say that a number M is the *algebraic degree of precision* (ADP) of (3.1) if (3.1) is exact for all polynomials of degree M and there is a polynomial of degree $M+1$ for which this formula is not exact.

Formulas of type (3.1) have been investigated in [3]. Here we state one of the theorems derived in [3], which applies to our case.

Theorem 3.1 *The quadrature formula*

$$\int_{-1}^{1} \sqrt{1-t^2}U_n(t)f(t)\,dt \approx \sum_{j=1}^{n+1} a_j f\left(\cos\frac{(2j-1)\pi}{2n+2}\right), \tag{3.2}$$

with

$$a_j = (-1)^{j-1}\frac{\pi}{2n+2}\sin\frac{(2j-1)\pi}{2n+2}$$

is the unique formula of highest ADP (equal to $3n+1$*) among all formulas of this type with* $n+1$ *nodes.*

4 Quadratures for the Fourier-Chebyshev coefficients

In this section, we consider formulas of type

$$\int_B f(x,y)U_n(x\cos\theta + y\sin\theta)\,dxdy \approx \sum_{j=1}^{n+1} b_j \mathcal{R}(f;\xi_j,\theta), \tag{4.1}$$

with nodes $\{\xi_j\}$ and coefficients $\{b_j\}$. Clearly (4.1) is not exact for the polynomial

$$U_n(x\cos\theta + y\sin\theta)\prod_{j=1}^{n+1}(x\cos\theta + y\sin\theta - \xi_j)^2,$$

and therefore ADP(4.1)$\leq 3n+1$. Formulas of type (4.1) with ADP$= 3n+1$ are called *Gaussian*. The following theorem holds.

Theorem 4.1 *There is a unique Gaussian quadrature of type* (4.1)*, given by*

$$\int_B f(x,y)U_n(x\cos\theta + y\sin\theta)\,dxdy \approx \frac{\pi}{2n+2}\sum_{j=1}^{n+1}(-1)^{j-1}\mathcal{R}\left(f; cos\frac{(2j-1)\pi}{2n+2},\theta\right). \tag{4.2}$$

Proof: For every angle θ and function G, we have

$$\int_B G(x,y)\,dxdy = \int_{-1}^{1}\left[\int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} G(t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta)\,ds\right]dt,$$

and thus

$$\begin{aligned}
&\int_B f(x,y)U_n\left(x\cos\theta + y\sin\theta\right)\,dxdy \\
&= \frac{1}{\pi}\int_{-1}^{1}\left[\int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} f(t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta)\,ds\right]U_n(t)\,dt \\
&= \frac{1}{\pi}\int_{-1}^{1}\mathcal{R}(f;t,\theta)U_n(t)\,dt. \qquad (4.3)
\end{aligned}$$

Consider a formula of type (4.1) with coefficients $\{b_j\}$ and nodes $\{\xi_j\}$. Note that for every bivariate polynomial Q of degree $3n+1$, $\mathcal{R}(Q;t,\theta) = \sqrt{1-t^2}Q_\theta(t)$, where Q_θ is a polynomial of degree $3n+1$ whose coefficients are trigonometric polynomials in θ. Also, all univariate polynomials of degree $3n+1$ could be described as Q_θ for some $Q \in \pi_{3n+1}(I\!R^2)$. From this observation and (4.3), it follows that

$$\begin{aligned}
\int_B Q(x,y)U_n(x\cos\theta + y\sin\theta)\,dxdy &= \int_{-1}^{1}\mathcal{R}(Q;t,\theta)U_n(t)\,dt \\
&= \int_{-1}^{1}\sqrt{1-t^2}U_n(t)Q_\theta(t)\,dt,
\end{aligned}$$

and

$$\sum_{j=1}^{n+1} b_j\mathcal{R}(Q;\xi_j,\theta) = \sum_{j=1}^{n+1} b_j\sqrt{1-\xi_j^2}Q_\theta(\xi_j).$$

Therefore formula (4.1) is Gaussian if and only if the formula

$$\int_{-1}^{1}\sqrt{1-t^2}U_n(t)Q_\theta(t)\,dt = \sum_{j=1}^{n+1} b_j\sqrt{1-\xi_j^2}Q_\theta(\xi_j) \qquad (4.4)$$

is exact for all polynomials $Q_\theta \in \pi_{3n+1}(I\!R)$. Now we apply Theorem 3.1 and derive that $\xi_j = \cos\frac{(2j-1)\pi}{2n+2}$, $j = 1,\ldots,n+1$, and that the coefficients b_j are given by

$$b_j = (-1)^{j-1}\frac{\pi}{2n+2}\sin\frac{(2j-1)\pi}{2n+2}\frac{1}{\sqrt{1-\xi_j^2}} = (-1)^{j-1}\frac{\pi}{2n+2}, \qquad j = 1,\ldots,n+1.$$

The proof is completed. □

Now, let us return to the computation of the coefficients $c_{k,n}(f)$. We apply the Gaussian quadrature (4.2) from Theorem 4.1 and derive that

$$c_{k,n}(f) \approx \frac{1}{2n+2}\sum_{j=1}^{n+1}(-1)^{j-1}\mathcal{R}\left(f;\cos\frac{(2j-1)\pi}{2n+2},\theta_{k,n}\right). \qquad (4.5)$$

This formula computes exactly the coefficients $c_{k,n}(Q)$ of a bivariate polynomial Q of degree $3n+1$, using the $n+1$ Radon projections along the line segments $I(\cos\frac{(2j-1)\pi}{2n+2}, \theta_{k,n})$, $j = 1, \ldots, n+1$. Notice that the calculation of $c_{k,n}(f)$ does not involve multiplication but only the addition/subtraction of the corresponding Radon transforms, which, in addition to the improved accuracy, improves the computational time and memory efficiency of the proposed formula.

A related question, which is still open, is whether formula (4.2) is the only Gaussian formula among formulas of type

$$\int_B f(x,y)U_n(x\cos\theta + y\sin\theta)\,dxdy \approx \sum_{j=1}^{n+1} b_j \mathcal{R}(f;\xi_j,\theta_j),$$

where the Radon transforms are taken not along parallel lines, but any $n+1$ lines $I(\xi_j, \theta_j)$ in the ball B .

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