Journal of Computational and Applied Mathematics 290 (2015) 209–223

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Extended Gaussian type cubatures for the ball

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ARTICLE INFO

Article history:
Received 10 July 2014
Received in revised form 18 February 2015

MSC:
65D32
65D30
41A55

Keywords:
Extended cubature formulae
Polyharmonic functions
Polyharmonic degree of precision

ABSTRACT

We construct cubatures that approximate the integral of a function u over the unit ball by the linear combination of surface integrals over the unit sphere of normal derivatives of u and surface integrals of u and $\Delta^2 u$ over m spheres, centered at the origin. We derive explicitly the weights and the nodes of these cubatures, and show that they are exact for all $(2m+2)$-harmonic functions.

1. Introduction

Recently, there has been a substantial effort to extend some of the classical results on quadrature formulas to higher dimensions. New cubature formulas for balls, simplices, spheres and parallelepipeds, based on integrals over low dimensional manifolds, have been suggested, see [1–7], and the references therein. The interest in such cubatures stems from various new technological advances and the need for rigorous mathematical theory to support them. For example, the recent development of the Thermoacoustic Tomography (TT) as one of the promising methods of medical imaging revived the interest in the so-called circular Radon transform, which integrates a function over a set of spheres with a given set of centers. The TT procedure sends a short microwave or radio-frequency electromagnetic pulses through a biological object. At each internal location $\mathbf{x}$, certain energy $u(\mathbf{x})$ is absorbed. The absorbed energy, due to resulting heating, causes thermoelastic expansion, which in turn creates a pressure wave. This wave can be detected by ultrasound transducers placed outside the body. It has been shown that these transducers effectively measure the integrals of u over all spheres centered at the transducers locations. The development of this new technology requires answers to several associated with it mathematical problems, among which are the uniqueness, stability and efficiency in the recovery of u or linear functionals of u from the given data.

In this paper, we deal with the construction and the characterization of several cubatures that approximate the integral of a function u over the unit ball by the linear combination of surface integrals of u over spheres, centered at the origin. More precisely, we construct high dimensional analogues to the result from [8], where Turan's problem [9] of finding a quadrature formula of the form

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{k=1}^{m} (a_k f(x_k) + b_k f''(x_k)), \tag{1.1}$$

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http://dx.doi.org/10.1016/j.cam.2015.05.010
0377-0427/

that is exact for all univariate polynomials of degree at least $2m$ has been solved. It was shown in [8] that the quadrature formula

$$\int_{-1}^{1} f(x)\,dx \approx \frac{2}{m(m+3)} \sum_{k=1}^{m} \left(\frac{f(x_k)}{P_{m+1}^2(x_k)} + \frac{(1-x_k^2)f''(x_k)}{(m+1)(m+2)P_{m+1}^2(x_k)} \right),$$

where P_{m+1} is the Legendre polynomial and $\{x_i\}_{i=1}^m$ are the zeroes of P'_{m+1}, has algebraic degree of precision $2m+1$. This result has been extended in [10], where the authors present a cubature for approximating the integral of a function u over the unit ball $B := \{\mathbf{x} \in \mathbb{R}^n : \ \|\mathbf{x}\| := (\sum_{i=1}^n x_i^2)^{1/2} < 1\}$ in $\mathbb{R}^n$ using, instead of point evaluations, integrals over spheres $S(r_k) := \{\mathbf{x} \in \mathbb{R}^n : \ \|\mathbf{x}\| = r_k\}$, namely

$$\int_B u(\mathbf{x})\,d\mathbf{x} \approx \sum_{k=1}^{m} \left(A_k \int_{S(r_k)} u(\xi)\,d\sigma(\xi) + B_k \int_{S(r_k)} \Delta u(\xi)\,d\sigma(\xi) \right). \tag{1.2}$$

They showed that formula (1.2) is exact for all polyharmonic functions of order $2m+1$, and explicitly computed the weights $\{A_k\}$, $\{B_k\}$ and the nodes $\{r_k\}$.

Here, we present analogues to cubature (1.2), where we use information along the boundary $S(1)$ of the integration domain B and integrals over spheres $\{S(\tau_j)\}$ of the function and its second order Laplacian Δ^2. To further describe our results, we need some notation. Let us denote by $B(r)$ the Euclidean ball in $\mathbb{R}^n$ with radius r. Recall that a function u, defined on B, is called a *polyharmonic function of order* p (or p-harmonic function), see [11,12], if $u \in C^{2p-1}(\bar{B}) \cap C^{2p}(B)$ and it satisfies the equation

$$\Delta^p u(\mathbf{x}) = 0, \quad \mathbf{x} \in B, \text{ where } \Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}, \qquad \Delta^p := \Delta(\Delta^{p-1}). \tag{1.3}$$

In particular, when $p = 1$ ($p = 2$), u is called harmonic (biharmonic). The set of all p-harmonic functions on B is denoted by $H^p(B)$. We also denote by $\frac{\partial u}{\partial \nu}$ the normal derivative of u, where ν is the outward unit normal to the sphere $S(1)$.

In this paper, we construct cubature of type I, that is cubature of the form

$$\int_B u(\mathbf{x})\,d\mathbf{x} \approx A \int_{S(1)} u(\xi)d\sigma(\xi) + \sum_{j=1}^{m} \left(B_j \int_{S(\tau_j)} u(\xi)\,d\sigma(\xi) + C_j \int_{S(\tau_j)} \Delta^2 u(\xi)\,d\sigma(\xi) \right), \tag{1.4}$$

and cubature of type II,

$$\int_B u(\mathbf{x})\,d\mathbf{x} \approx F \int_{S(1)} \frac{\partial u}{\partial \nu}(\xi)\,d\sigma(\xi) + \sum_{j=1}^{m} \left(G_j \int_{S(\tau_j)} u(\xi)\,d\sigma(\xi) + H_j \int_{S(\tau_j)} \Delta^2 u(\xi)\,d\sigma(\xi) \right), \tag{1.5}$$

that are exact for all $(2m+2)$-harmonic functions. We call such formulas Lobatto–Turan cubatures. We view them as multidimensional analogues of the classical Lobatto quadratures since they use information along the boundary $S(1)$ of the integration domain B, such as $\int_{S(1)} \frac{\partial u}{\partial \nu}(\xi)\,d\sigma(\xi)$ or $\int_{S(1)} u(\xi)\,d\sigma(\xi)$. They are also multidimensional generalizations of Turan's problem (1.1), because they involve higher order derivatives of the integrand, that is $\left\{\int_{S(\tau_j)} \Delta^2 u(\xi)\,d\sigma(\xi)\right\}_j$.

Since all polynomials in n variables of degree at most $2p-1$ are p-harmonic functions, namely

$$\pi_{2p-1}(\mathbb{R}^n) \subset H^p(B), \tag{1.6}$$

finding multi-dimensional cubature formulas that are exact for $H^p(B)$ for p as large as possible is a natural generalization of the notion of Gaussian quadratures in the one dimensional case. The largest natural number ℓ for which a cubature is exact for all $u \in H^\ell(B)$ is called a *Polyharmonic Degree of Precision* (PDP) of this cubature. Formulas for numerical integration with the best possible PDP are called Gaussian cubatures. In the process of deriving cubatures (1.4) and (1.5), we obtain also a formula of the form

$$\int_B u(\mathbf{x})\,d\mathbf{x} \approx E_0 \int_{S(1)} u(\xi)d\sigma(\xi) + E_1 \int_{S(1)} \frac{\partial}{\partial \nu} u(\xi)\,d\sigma(\xi) + \sum_{j=1}^{m} D_j \int_{S(\tau_j)} u(\xi)\,d\sigma(\xi), \tag{1.7}$$

that has PDP(1.7) $= 2m+2$. We call it the Gauss–Lobatto cubature for the ball because it is a natural generalization of a one dimensional Gauss–Lobatto quadrature, and we show that there are no other cubatures of this form that integrate exactly polyharmonic functions of higher order. We also explicitly construct a cubature of the type

$$\int_B u(\mathbf{x})\,d\mathbf{x} \approx P_0 \int_{S(1)} u(\xi)d\sigma(\xi) + P_1 \int_{S(1)} \frac{\partial}{\partial \nu} u(\xi)\,d\sigma(\xi) + \sum_{j=1}^{m} Q_j \int_{S(\tau_j)} \Delta^2 u(\xi)\,d\sigma(\xi), \tag{1.8}$$

exact for all $u \in H^{2m+2}(B)$, which we call the Gauss–Lobatto–Turan cubature for the ball. We show the uniqueness of this formula and prove that no other cubature of this type can be exact for polyharmonic functions of higher order.

The paper is organized as follows. In Section 2, we provide several results, needed for the construction of the above cubatures. The computation of the weights and the nodes of a classical quadrature, used in the construction of the new formulas is performed in Section 3. We obtain a new Gauss–Lobatto cubature for the ball in Section 4, and a new Gauss–Lobatto–Turan cubature for the ball in Section 5. The Lobatto–Turan cubatures of type I and type II are described in Section 6.

2. Preliminaries

In this section, we discuss some results, needed later in the paper. First, we state a well known fact about the representation of p-harmonic functions, see [13, Lemma 2].

Lemma 2.1. *For each $u \in H^p(B)$, there exist unique functions $b_0, \dots, b_{p-1}$, each harmonic on B, such that*

$$u(\mathbf{x}) = \sum_{j=0}^{p-1} |\mathbf{x}|^{2j} b_j(\mathbf{x}), \quad \mathbf{x} \in B. \tag{2.1}$$

Conversely, each function of the form (2.1), where $b_0, \dots, b_{p-1}$ are harmonic functions, is a p-harmonic function.

Proof. Formula (2.1) is the classical Almansi's expansion of polyharmonic functions, see [11, Proposition 1.3, p. 4]. To prove the rest of the lemma, we will first show by induction that for every harmonic function b,

$$\Delta^m \left[b(\mathbf{x})|\mathbf{x}|^{2(m-1)}\right] = 0, \quad \mathbf{x} \in B. \tag{2.2}$$

Clearly, $\Delta b = 0$, since b is harmonic. Let us assume that $\Delta^{m-1}\left[b(\mathbf{x})|\mathbf{x}|^{2(m-2)}\right] = 0$, $\mathbf{x} \in B$, for any harmonic function b. Then, we have

$$\begin{aligned} \Delta^m \left[b(\mathbf{x})|\mathbf{x}|^{2(m-1)}\right] &= \Delta^{m-1}\left[\Delta(b(\mathbf{x})|\mathbf{x}|^{2(m-1)})\right] \\ &= 2(m-1)(n+2m-4)\Delta^{m-1}\left[b(\mathbf{x})|\mathbf{x}|^{2(m-2)}\right] + \Delta^{m-1}\left[|\mathbf{x}|^{2(m-1)}\Delta b\right] \\ &\quad + 4(m-1)\Delta^{m-1}\left[|\mathbf{x}|^{2(m-2)} \sum_{i=1}^{n} x_i \frac{\partial b}{\partial x_i}\right] \\ &= 4(m-1)\Delta^{m-1}\left[|\mathbf{x}|^{2(m-2)} \sum_{i=1}^{n} x_i \frac{\partial b}{\partial x_i}\right], \end{aligned} \tag{2.3}$$

where we have used the induction hypothesis and the fact that b is harmonic. Notice that the function $\sum_{i=1}^{n} x_i \frac{\partial b}{\partial x_i}$ is harmonic as well, since

$$\Delta\left[\sum_{i=1}^{n} x_i \frac{\partial b}{\partial x_i}\right] = \sum_{i=1}^{n} \Delta\left[x_i \frac{\partial b}{\partial x_i}\right] = 2\Delta b + \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}(\Delta b) = 0,$$

and therefore the last expression in (2.3) is also equal to zero because of the induction hypothesis. This proves (2.2). Now, if u has the representation (2.1), using (2.2), we derive that

$$\Delta^p u(\mathbf{x}) = \sum_{j=0}^{p-1} \Delta^p\left[|\mathbf{x}|^{2j} b_j(\mathbf{x})\right] = \sum_{j=0}^{p-1} \Delta^{p-j-1}\left[\Delta^{j+1}\left[|\mathbf{x}|^{2j} b_j(\mathbf{x})\right]\right] = 0,$$

and the proof is completed. ■

The next result is a simple fact about harmonic functions.

Lemma 2.2. *Let b be a harmonic function on B and $0 < r \le 1$. Then we have*

$$\int_{S(r)} b(\mathbf{x})\, d\mathbf{x} = \gamma_n r^{n-1} b(0), \quad \gamma_n := \frac{n\pi^{n/2}}{\Gamma(n/2+1)}, \tag{2.4}$$

and

$$\int_{S(r)} \frac{\partial b}{\partial \nu}(\xi)\, d\sigma(\xi) = 0. \tag{2.5}$$

3. An auxiliary quadrature

In this section, we compute the weights and nodes of the quadrature formula

$$\int_{-1}^{1}(1+x)^{\frac{n}{2}-1}f(x)\,dx \approx e_0f(1)+e_1f'(1)+\sum_{j=1}^{m}d_jf(x_j), \tag{3.1}$$

which we use to construct the cubatures of type I and type II. The existence and uniqueness of such formula with highest algebraic degree of precision equal to $2m+1$ is a classical result. Its nodes $x_1,\ldots,x_m$ are the zeroes of the Jacobi polynomials $P_m^{(2,\frac{n}{2}-1)}$. These polynomials are orthogonal on $(-1,1)$ with respect to the weight function $(1-x)^2(1+x)^{n/2-1}$. In the notation used here, they are normalized such that $P_m^{(2,\frac{n}{2}-1)}(1)=\frac{1}{2}(m+2)(m+1)$.
If we use the Pochhammer symbol, defined by

$$(a)_0=1,\qquad (a)_j=a(a+1)\cdots(a+j-1)=\frac{\Gamma(a+j)}{\Gamma(a)},\quad j=1,2,\ldots,$$

with Γ being the Gamma function, we recall that the generalized hypergeometric series is given by

$${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z)=\sum_{j=0}^{\infty}\frac{(a_1)_j\ldots(a_p)_j}{(b_1)_j\ldots(b_q)_j}\frac{z^j}{j!}.$$

Note that Jacobi polynomials $P_m^{(2,\frac{n}{2}-1)}$ are hypergeometric polynomials, namely that

$$\begin{aligned}P_m^{(2,\frac{n}{2}-1)}(x) &= \frac{1}{2}(m+1)(m+2)\,{}_2F_1(-m,m+n/2+2;3;(1-x)/2)\\ &= \frac{1}{2}(m+1)(m+2)\sum_{j=0}^{m}\frac{(-m)_j(m+n/2+2)_j}{(3)_j}\frac{(1-x)^j}{2^jj!}.\end{aligned} \tag{3.2}$$

For simplicity, we denote by

$$R(x):={}_2F_1(-m,m+n/2+2;3;(1-x)/2).$$

Clearly, $R(1)=1$. Further, we will need the following technical lemma which explicitly computes the integral below.

Lemma 3.1.

$$I(p):=\int_{-1}^{1}(1+x)^{\frac{n}{2}-1}(1-x)^pR(x)\,dx=\frac{2^{n/2+p}p!(n/2)_m(2-p)_m}{(n/2)_{p+1}(3)_m(n/2+p+1)_m}.$$

Proof. We use the change of variables $1+x=2t$ to derive

$$\begin{aligned}I(p) &= 2^{n/2+p}\int_0^1 t^{n/2-1}(1-t)^p{}_2F_1(-m,m+n/2+2;3;1-t)dt\\ &= 2^{n/2+p}\sum_{j=0}^{m}\frac{(-m)_j(m+n/2+2)_j}{(3)_jj!}\int_0^1 t^{n/2-1}(1-t)^{j+p}\,dt\\ &= 2^{n/2+p}\sum_{j=0}^{m}\frac{(-m)_j(m+n/2+2)_j(j+p)!}{(3)_jj!}\frac{\Gamma(n/2)}{\Gamma(n/2+j+p+1)}\\ &= \frac{2^{n/2+p}p!}{(n/2)_{p+1}}\sum_{j=0}^{m}\frac{(-m)_j(m+n/2+2)_j(p+1)_j}{(3)_j(n/2+p+1)_j}\frac{1}{j!}\\ &= \frac{2^{n/2+p}p!}{(n/2)_{p+1}}\cdot{}_3F_2(-m,m+n/2+2,p+1;3,n/2+p+1;1).\end{aligned}$$

Application of Saalschütz's formula, see [14, p. 9, Eq. (1)],

$${}_3F_2(-m,a,b;c,1+a+b-c-m;1)=\frac{(c-a)_m(c-b)_m}{(c)_m(c-a-b)_m}, \tag{3.3}$$

for $a=m+n/2+2,\ b=p+1,\ c=3$, gives

$$I(p)=\frac{2^{n/2+p}p!(n/2)_m(2-p)_m}{(n/2)_{p+1}(3)_m(n/2+p+1)_m},$$

and the proof is completed. ■

The next lemma provides explicitly the nodes and weights of quadrature (3.1).

Lemma 3.2. *The quadrature formula (3.1) with weights*

$$e_0 = 2^{\frac{n}{2}+2}\frac{(8m^2+4mn+16m+3n+6)}{3(m+1)(m+2)(n+2m)(n+2m+2)},$$

$$e_1 = -2^{n/2+4}\frac{1}{(m+1)(m+2)(n+2m)(n+2m+2)},$$

$$d_j = 2^{n/2+2}\frac{(4m+n+6)^2(m+2)}{(m+1)(2m+n+4)^2(2m+n)(2m+n+2)}\cdot\frac{(1+x_j)}{(1-x_j)\left[P_{m+1}^{(2,n/2-1)}(x_j)\right]^2},$$

and nodes $\{x_j\}_{j=1}^m$ *the zeroes of the Jacobi polynomials* $P_m^{(2,n/2-1)}$ *is the only quadrature of this form that has ADP(3.1)* $= 2m+1$.

Proof. Since the existence and uniqueness of this quadrature is a classical result, we shall simply compute its weights. First, we compute e_0 and e_1. We apply (3.1) to $(1-x)R(x) \in \pi_{m+1}(\mathbb{R})$. Using Lemma 3.1, we have

$$e_1 = -I(1) = -\frac{2^{n/2+1}m!}{(3)_m(n/2+m)_2} = -\frac{2^{n/2+4}}{(m+1)(m+2)(n+2m)(n+2m+2)}.$$

To calculate the weight e_0, we apply (3.1) to the polynomial R and derive that

$$e_0R(1) + e_1R'(1) = I(0).$$

Since

$$R'(1) = \frac{2}{(m+1)(m+2)}\frac{d}{dx}P_m^{(2,\frac{n}{2}-1)}(1) = \frac{2m+n+4}{2(m+1)(m+2)}P_{m-1}^{(3,\frac{n}{2})}(1) = \frac{m(2m+n+4)}{12},$$

and from Lemma 3.1, we have

$$I(0) = \frac{2^{\frac{n}{2}+2}}{(m+2)(n+2m)},$$

we obtain

$$e_0 = I(0) - e_1R'(1) = \frac{2^{\frac{n}{2}+2}(8m^2+4mn+16m+3n+6)}{3(m+1)(m+2)(n+2m)(n+2m+2)}.$$

Next, to calculate the coefficients $\{d_j\}_{j=1}^m$, we apply (3.1) to $(1-x)^2S(x)$, where S is any polynomial from $\pi_{2m-1}(\mathbb{R})$. It follows that

$$\int_{-1}^{1}(1+x)^{n/2-1}(1-x)^2S(x)\,dx = \sum_{j=1}^{m}d_j(1-x_j)^2S(x_j),$$

and therefore this is the unique Gaussian quadrature on $(-1, 1)$ with the weight $(1+x)^{n/2-1}(1-x)^2$. The nodes of this formula are the zeroes of $P_m^{(2,n/2-1)}$ and the coefficients λ_j are given by, see [15, p. 352, Eq. (15.3.1)],

$$\lambda_j = 2^{n/2+2}\frac{4(m+1)(m+2)}{(2m+n)(2m+n+2)}\frac{1}{(1-x_j^2)}\left\{\frac{d}{dx}P_m^{(2,n/2-1)}(x_j)\right\}^{-2}$$
$$= 2^{n/2+2}\frac{(4m+n+6)^2(m+2)}{(m+1)(2m+n+4)^2(2m+n)(2m+n+2)}\cdot\frac{(1-x_j^2)}{\left[P_{m+1}^{(2,n/2-1)}(x_j)\right]^2},$$

where in the last equality we have used the fact, see [15, (4.5.7)],

$$\frac{d}{dx}P_m^{(2,n/2-1)}(x_j) = -\frac{2(m+1)(2m+n+4)}{4m+n+6}\frac{P_{m+1}^{(2,n/2-1)}(x_j)}{(1-x_j^2)}.$$

Therefore, we have

$$d_j = \lambda_j(1-x_j)^{-2}$$
$$= 2^{n/2+2}\frac{(4m+n+6)^2(m+2)}{(m+1)(2m+n+4)^2(2m+n)(2m+n+2)}\cdot\frac{(1+x_j)}{(1-x_j)\left[P_{m+1}^{(2,n/2-1)}(x_j)\right]^2},$$

and the proof is completed. ■

Next, we provide several tables with the weights and nodes, see Tables 1–7, of the quadrature

$$\int_{-1}^{1}(1+x)^{\frac{1}{2}}f(x)\,dx \approx e_0 f(1)+e_1 f'(1)+\sum_{j=1}^{m} d_j f(x_j), \tag{3.4}$$

which is (3.1) for $n=3$.

Table 1
Formula (3.4), $m=1$, $e_0=0.915872$, $e_1=-0.215499$.

j	1
d_j	0.969746
x_j	−0.333333

Table 2
Formula (3.4), $m=2$, $e_0=0.513806$, $e_1=-0.059861$.

j	1	2
d_j	0.396975	0.974837
x_j	−0.645661	0.184123

Table 3
Formula (3.4), $m=3$, $e_0=0.325698$, $e_1=-0.022856$.

j	1	2	3
d_j	0.198750	0.593721	0.767450
x_j	−0.780044	−0.212793	0.463425

Table 4
Formula (3.4), $m=4$, $e_0=0.224164$, $e_1=-0.010549$.

j	1	2	3	4
d_j	0.113076	0.372180	0.588545	0.587653
x_j	−0.850186	−0.444336	0.099507	0.623586

Table 5
Formula (3.4), $m=5$, $e_0=0.163466$, $e_1=-0.005526$.

j	1	2	3	4	5
d_j	0.070277	0.244448	0.429905	0.522122	0.455400
x_j	−0.891404	−0.588710	−0.156285	0.313970	0.722429

Table 6
Formula (3.4), $m=6$, $e_0=0.124387$, $e_1=-0.003169$.

j	1	2	3	4	5	6
d_j	0.046593	0.167766	0.314448	0.424946	0.447498	0.359981
x_j	−0.917675	−0.684043	−0.336962	0.067332	0.463376	0.787282

Table 7
Formula (3.4), $m=7$, $e_0=0.097784$, $e_1=-0.001946$.

j	1	2	3	4	5	6	7
d_j	0.032450	0.119572	0.233610	0.337241	0.394313	0.380360	0.290288
x_j	−0.935446	−0.750014	−0.467351	−0.123498	0.237711	0.570260	0.831975

4. Gauss–Lobatto cubature for the ball

In this section, we investigate cubatures of type (1.7) that have maximal possible Polyharmonic Degree of Precision. Clearly, the PDP(1.7) $\le 2m+2$, since it is not exact for the polynomial

$$\left(1-|\mathbf{x}|^2\right)^2\left(|\mathbf{x}|^2-\tau_1^2\right)^2\cdots\left(|\mathbf{x}|^2-\tau_m^2\right)^2 \in \pi_{4m+4}(\mathbb{R}^n).$$

The existence and uniqueness of formulas of type (1.7) is closely related to the existence and uniqueness of quadratures of type (3.1), as the following lemma, similar to Lemma 3 in [13], claims.

Lemma 4.1. *Let μ be a weight function on $(0, 1)$ and $0 < \tau_1 < \cdots < \tau_m < 1$. Then cubature*

$$\int_B \mu(|\mathbf{x}|)u(\mathbf{x})\,d\mathbf{x} \approx \widetilde{E}_0 \int_{S(1)} u(\xi)d\sigma(\xi) + \widetilde{E}_1 \int_{S(1)} \frac{\partial}{\partial\nu}u(\xi)\,d\sigma(\xi) + \sum_{j=1}^{N} \widetilde{D}_j \int_{S(\tau_j)} u(\xi)\,d\sigma(\xi),$$

is exact for $H^p(B)$ if and only if the quadrature

$$\int_0^1 \mu(t)t^{n-1}f(t^2)\,dt \approx \widetilde{E}_0 f(1) + 2\widetilde{E}_1 f'(1) + \sum_{j=1}^{N} \widetilde{D}_j \tau_j^{n-1} f\left(\tau_j^2\right), \tag{4.1}$$

is exact for all polynomials in $\pi_{p-1}(\mathbb{R})$.

Proof. First, let us observe a relation between $H^p(B)$ and the polynomial space $\pi_{p-1}(\mathbb{R})$. We can associate to every p-harmonic function u, which according to Lemma 2.1 has the representation

$$u(\mathbf{x}) = \sum_{j=0}^{p-1} |\mathbf{x}|^{2j} b_j(\mathbf{x}), \tag{4.2}$$

a polynomial P, defined as

$$P(t) := \sum_{j=0}^{p-1} t^j b_j(0) \in \pi_{p-1}(\mathbb{R}). \tag{4.3}$$

Note that, because of Lemma 2.1, every $P \in \pi_{p-1}(\mathbb{R})$ has the form (4.3), where the b_j's come from the representation of some p-harmonic function u. Next, we integrate (4.2) over the sphere $S(t)$ and obtain

$$\begin{aligned}\int_{S(t)} u(\xi)\,d\sigma(\xi) &= \sum_{j=0}^{p-1} t^{2j} \int_{S(t)} b_j(\xi)\,d\sigma(\xi) = \gamma_n t^{n-1} \sum_{j=0}^{p-1} t^{2j} b_j(0) \\ &= \gamma_n t^{n-1} P(t^2),\end{aligned} \tag{4.4}$$

where we have used Lemma 2.2 in the next to the last equality. We also compute the normal derivative of u,

$$\frac{\partial}{\partial\nu}u(\mathbf{x}) = \sum_{j=0}^{p-1} |\mathbf{x}|^{2j} \frac{\partial}{\partial\nu} b_j(\mathbf{x}) + \sum_{j=1}^{p-1} 2j|\mathbf{x}|^{2j-1} b_j(\mathbf{x}),$$

and again by integrating over $S(t)$ and using Lemma 2.2, we get

$$\begin{aligned}\int_{S(t)} \frac{\partial}{\partial\nu}u(\xi)\,d\sigma(\xi) &= \sum_{j=0}^{p-1} t^{2j} \int_{S(t)} \frac{\partial}{\partial\nu} b_j(\xi)\,d\sigma(\xi) + \sum_{j=1}^{p-1} 2jt^{2j-1} \int_{S(t)} b_j(\xi)\,d\sigma(\xi) \\ &= \gamma_n t^{n-1} \sum_{j=1}^{p-1} 2jt^{2j-1} b_j(0) = \gamma_n t^{n-1} \frac{d}{dt}\left[P(t^2)\right] = 2\gamma_n t^n P'(t^2).\end{aligned} \tag{4.5}$$

Change of variables and (4.4) give

$$\int_B \mu(|\mathbf{x}|)u(\mathbf{x})\,d\mathbf{x} = \int_0^1 \int_{S(t)} \mu(|\xi|)u(\xi)\,d\sigma(\xi)dt = \gamma_n \int_0^1 \mu(t)t^{n-1}P(t^2)\,dt. \tag{4.6}$$

On the other hand, applying (4.4) and (4.5), we derive

$$\begin{aligned}&\widetilde{E}_0 \int_{S(1)} u(\xi)d\sigma(\xi) + \widetilde{E}_1 \int_{S(1)} \frac{\partial}{\partial\nu}u(\xi)\,d\sigma(\xi) + \sum_{j=1}^{N} \widetilde{D}_j \int_{S(\tau_j)} u(\xi)\,d\sigma(\xi) \\ &= \gamma_n \widetilde{E}_0 P(1) + 2\gamma_n \widetilde{E}_1 P'(1) + \gamma_n \sum_{j=1}^{N} \widetilde{D}_j \tau_j^{n-1} P(\tau_j^2).\end{aligned} \tag{4.7}$$

Comparison of (4.6) and (4.7) and the relation between p-harmonic functions and polynomials from $\pi_{p-1}(\mathbb{R})$ completes the proof of the lemma. ■

Table 8
Formula (1.7), $m = 1$, $E_0 = 0.203704$, $E_1 = -0.013889$.

j	1
D_j	0.592593
τ_j	0.500000

Table 9
Formula (1.7), $m = 2$, $E_0 = 0.106481$, $E_1 = -0.003472$.

j	1	2
D_j	0.431427	0.328753
τ_j	0.350021	0.737666

Application of Lemmas 4.1 and 3.2 lead to the proof of the existence and uniqueness of the Gauss–Lobatto formula for the ball, stated in the next theorem.

Theorem 4.2. *There is a unique Gauss–Lobatto cubature for the ball of type (1.7) with PDE(1.7) $= 2m + 2$. Its nodes are given by $\tau_j = \sqrt{(1 + x_j)/2}$, where $\{x_j\}_{j=1}^m$ are the zeros of the Jacobi polynomial $P_m^{(2,\frac{n}{2}-1)}$, and its weights are*

$$E_0 = 2^{-\frac{n}{2}-1}e_0, \qquad E_1 = 2^{-\frac{n}{2}-3}e_1, \qquad D_j = 2^{-\frac{3}{2}}(x_j + 1)^{-\frac{n}{2}+\frac{1}{2}}d_j, \quad j = 1, \dots, m,$$

where e_0, e_1 and $\{d_j\}_{j=1}^m$ are the coefficients from Lemma 3.2.

Proof. It follows from Lemma 4.1 with $\mu \equiv 1$, $N = m$, and $p = 2m + 2$, that formula (1.7) is the only cubature with PDP$= 2m + 2$ if and only if

$$\int_0^1 t^{n-1}f(t^2)\,dt \approx E_0f(1) + 2E_1f'(1) + \sum_{j=1}^m D_j\tau_j^{n-1}f\left(\tau_j^2\right)$$

is the only quadrature, exact for all polynomials from $\pi_{2m+1}(\mathbb{R})$. After change of variables $1+x = 2t^2$ and the new notation,

$$g(x) := f\left(\frac{1+x}{2}\right)$$

the above quadrature can be rewritten as

$$\int_{-1}^1 (1+x)^{\frac{n}{2}-1}g(x)\,dx \approx 2^{\frac{n}{2}+1}E_0g(1) + 2^{\frac{n}{2}+3}E_1g'(1) + 2^{\frac{n}{2}+1}\sum_{j=1}^m D_j\tau_j^{n-1}g\left(2\tau_j^2 - 1\right).$$

We apply Lemma 3.2 and derive that

$$E_0 = 2^{-\frac{n}{2}-1}e_0, \qquad E_1 = 2^{-\frac{n}{2}-3}e_1, \qquad D_j = 2^{-\frac{n}{2}-1}\tau_j^{1-n}d_j, \quad \tau_j = \sqrt{(1+x_j)/2}. \quad \blacksquare$$

Remark 4.3. The weights from Theorem 4.2 can be computed explicitly and are

$$E_0 = \frac{2(8m^2 + 4mn + 16m + 3n + 6)}{3(m+1)(m+2)(n+2m)(n+2m+2)},$$
$$E_1 = -\frac{2}{(m+1)(m+2)(n+2m)(n+2m+2)},$$

and for $j = 1, \dots, m$,

$$D_j = 2^{\frac{n}{2}+\frac{1}{2}}\frac{(4m+n+6)^2(m+2)}{(m+1)(2m+n+4)^2(2m+n)(2m+n+2)} \cdot \frac{\left(1+x_j\right)^{-\frac{n}{2}+\frac{3}{2}}}{\left(1-x_j\right)\left[P_{m+1}^{(2,n/2-1)}(x_j)\right]^2},$$

where $\{x_j\}_{j=1}^m$ are the zeroes of the Jacobi polynomials $P_m^{(2,n/2-1)}$.

Next, we provide the coefficients and nodes of cubature (1.7), see Tables 8–14 in the two-dimensional case ($n = 2$).

Table 10
Formula (1.7), $m = 3$, $E_0 = 0.065000$, $E_1 = -0.001250$.

j	1	2	3
D_j	0.338406	0.291116	0.204948
τ_j	0.270174	0.589070	0.839644

Table 11
Formula (1.7), $m = 4$, $E_0 = 0.043704$, $E_1 = -0.000556$.

j	1	2	3	4
D_j	0.278077	0.253301	0.204613	0.139218
τ_j	0.220228	0.488468	0.719060	0.892105

Table 12
Formula (1.7), $m = 5$, $E_0 = 0.031368$, $E_1 = -0.000283$.

j	1	2	3	4	5
D_j	0.235883	0.221766	0.191887	0.150503	0.100503
τ_j	0.185954	0.416510	0.624409	0.796451	0.922526

Table 13
Formula (1.7), $m = 6$, $E_0 = 0.023597$, $E_1 = -0.000159$.

j	1	2	3	4	5	6
D_j	0.204752	0.196262	0.176714	0.149117	0.114937	0.075877
τ_j	0.160949	0.362717	0.549943	0.713468	0.845979	0.941704

Table 14
Formula (1.7), $m = 7$, $E_0 = 0.018390$, $E_1 = -0.000096$.

j	1	2	3	4	5	6	7
D_j	0.180852	0.175577	0.162143	0.142898	0.118710	0.090471	0.059276
τ_j	0.141890	0.321075	0.490463	0.643430	0.774611	0.879498	0.954559

5. Gauss–Lobatto–Turan cubature for the ball

In this section, we discuss cubatures of the form (1.8) that are exact for all functions $u \in H^{2m+2}(B)$. We start with a simple lemma.

Lemma 5.1. *If u has continuous Laplacians of second order $\Delta^2 u$ in a neighborhood of B, then the following formula holds:*

$$\int_B u(\mathbf{x})d\mathbf{x} = \frac{1}{n}\int_{S(1)} u(\xi)d\sigma(\xi) - \frac{1}{n(n+2)}\int_{S(1)} \frac{\partial}{\partial\nu}u(\xi)\, d\sigma(\xi) + \frac{1}{8n(n+2)}\int_B (1-|\mathbf{x}|^2)^2\Delta^2 u(\mathbf{x})\, d\mathbf{x}.$$

Proof. We apply the divergence theorem first to the vector field F,

$$F := (1-|\mathbf{x}|^2)^2\left(\frac{\partial}{\partial x_1}(\Delta u), \dots, \frac{\partial}{\partial x_n}(\Delta u)\right)$$

and obtain

$$\frac{1}{4}\int_B (1-|\mathbf{x}|^2)^2\Delta^2 u(\mathbf{x})\, d\mathbf{x} = \int_B (1-|\mathbf{x}|^2)\sum_{i=1}^n x_i\frac{\partial}{\partial x_i}(\Delta u)\ d\mathbf{x}, \tag{5.1}$$

and then to the vector field G,

$$G := (1-|\mathbf{x}|^2)\Delta u(x_1, \dots, x_n),$$

and derive

$$\int_B (1-|\mathbf{x}|^2)\sum_{i=1}^n x_i\frac{\partial}{\partial x_i}(\Delta u)\ d\mathbf{x} = 2\int_B \Delta u(\mathbf{x})\, d\mathbf{x} - (n+2)\int_B (1-|\mathbf{x}|^2)\Delta u(\mathbf{x})\, d\mathbf{x}.$$

It follows from the last two formulas that

$$\int_B (1-|\mathbf{x}|^2)^2\Delta^2 u(\mathbf{x})\, d\mathbf{x} = 8\int_B \Delta u(\mathbf{x})\, d\mathbf{x} - 4(n+2)\int_B (1-|\mathbf{x}|^2)\Delta u(\mathbf{x})\, d\mathbf{x}. \tag{5.2}$$

Similarly, we derive that

$$\int_B \Delta u(\mathbf{x})\, d\mathbf{x} = \int_{S(1)} \frac{\partial}{\partial \nu} u(\xi)\, d\sigma(\xi), \tag{5.3}$$

and

$$\int_B (1-|\mathbf{x}|^2)\Delta u(\mathbf{x})\, d\mathbf{x} = 2\int_{S(1)} u(\xi)\, d\sigma(\xi) - 2n\int_B u(\mathbf{x})\, d\mathbf{x}. \tag{5.4}$$

It follows from (5.2)–(5.4) that

$$\int_B (1-|\mathbf{x}|^2)^2 \Delta^2 u(\mathbf{x})\, d\mathbf{x} = 8n(n+2)\int_B u(\mathbf{x})\, d\mathbf{x} - 8(n+2)\int_{S(1)} u(\xi)\, d\sigma(\xi) + 8\int_{S(1)} \frac{\partial}{\partial \nu} u(\xi)\, d\sigma(\xi),$$

and the proof is completed. ■

Before deriving the Gauss–Lobatto–Turan formula for the ball, we will make several observations. The first one is the following lemma.

Lemma 5.2. *Every cubature formula of type (1.8) has PDP(1.8) $\leq 2m+2$.*

Proof. Let us assume that (1.8) is exact for all $u \in H^{2m+3}(B)$. In particular, it is exact for $u(\mathbf{x}) = 1$, and therefore the weight P_0 is uniquely determined from

$$P_0 \int_{S(1)} 1\, d\sigma(\xi) = \int_B 1\, d\mathbf{x},$$

which gives $P_0 = \frac{1}{n}$. Likewise, the formula should be exact for $u(\mathbf{x}) = |\mathbf{x}|^2$, namely we have

$$P_0 \int_{S(1)} |\xi|^2\, d\sigma(\xi) + P_1 \int_{S(1)} \frac{\partial}{\partial \nu}|\xi|^2\, d\sigma(\xi) = \int_B |\mathbf{x}|^2\, d\mathbf{x},$$

and therefore $P_1 = -\frac{1}{n(n+2)}$. Now, we use the formula from Lemma 5.1 and formula (1.8) with the so far determined weights P_0 and P_1 to derive that

$$\int_B (1-|\mathbf{x}|^2)^2 \Delta^2 u(\mathbf{x})\, d\mathbf{x} = 8n(n+2)\sum_{j=1}^{m} Q_j \int_{S(\tau_j)} \Delta^2 u(\xi)\, d\sigma(\xi), \tag{5.5}$$

for every $u \in H^{2m+3}(B)$. Let us select the function $\tilde{u}$, defined from the relation

$$\Delta^2\left[\tilde{u}(\mathbf{x})\right] := \prod_{i=1}^{m}(|\mathbf{x}|^2 - \tau_i^2)^2.$$

Clearly

$$\Delta^{2m+3}\left[\tilde{u}(\mathbf{x})\right] = \Delta^{2m+1}\left[\prod_{i=1}^{m}(|\mathbf{x}|^2 - \tau_i^2)^2\right] = 0,$$

which gives that $\tilde{u} \in H^{2m+3}(B)$. Therefore formula (5.5) should be exact for $\tilde{u}$. However, we have that

$$\int_B (1-|\mathbf{x}|^2)^2 \prod_{i=1}^{m}(|\mathbf{x}|^2 - \tau_i^2)^2\, d\mathbf{x} > 0, \quad \text{while} \sum_{j=1}^{m} Q_j \int_{S(\tau_j)} \prod_{i=1}^{m}(|\xi|^2 - \tau_i^2)^2\, d\sigma(\xi) = 0,$$

and thus PDP (1.8) $\leq 2m+2$. ■

Theorem 5.3. *There is a unique cubature of type (1.8), exact for all $u \in H^{2m+2}(B)$, with weights*

$$P_0 = \frac{1}{n}, \qquad P_1 = -\frac{1}{n(n+2)},$$

and for $j = 1, \ldots, m$,

$$Q_j = 2^{-\frac{13}{2}} \frac{(1-x_j)^2\left(1+x_j\right)^{-\frac{n}{2}+\frac{1}{2}}}{n(n+2)} d_j,$$

with $\{d_j\}_{j=1}^m$ *being the weights from* Lemma 3.2. *Its nodes are* $\tau_j = \sqrt{(1+x_j)/2}$, *where* $\{x_j\}_{j=1}^m$ *are the zeros of the Jacobi polynomial* $P_m^{(2,\frac{n}{2}-1)}$.

Proof. Let us first notice that, if $u \in H^{2m+2}(B)$, then the function

$$(1-|\mathbf{x}|^2)^2 \Delta^2 u(\mathbf{x}) \in H^{2m+2}(B),$$

see [6], and we can use the Gauss–Lobatto cubature (1.7), to compute the integral $\int_B (1-|\mathbf{x}|^2)^2 \Delta^2 u(\mathbf{x})\, d\mathbf{x}$. Since

$$\int_{S(1)} (1-|\xi|^2)^2 \Delta^2 u(\xi)\, d\sigma(\xi) = 0, \qquad \int_{S(1)} \frac{\partial}{\partial \nu} \left((1-|\xi|^2)^2 \Delta^2 u(\xi)\right) d\sigma(\xi) = 0,$$

we have

$$\int_B (1-|\mathbf{x}|^2)^2 \Delta^2 u(\mathbf{x})\, d\mathbf{x} = \sum_{j=1}^m D_j \left(1-\tau_j^2\right)^2 \int_{S(\tau_j)} \Delta^2 u(\xi)\, d\sigma(\xi).$$

The latter formula and Lemma 5.1 give the new cubature

$$\int_B u(\mathbf{x}) d\mathbf{x} = \frac{1}{n} \int_{S(1)} u(\xi) d\sigma(\xi) - \frac{1}{n(n+2)} \int_{S(1)} \frac{\partial}{\partial \nu} u(\xi)\, d\sigma(\xi) + \frac{1}{8n(n+2)} \sum_{j=1}^m D_j \left(1-\tau_j^2\right)^2 \int_{S(\tau_j)} \Delta^2 u(\xi)\, d\sigma(\xi).$$

Now, let us assume that (1.8) is another cubature with PDP(1.8) $= 2m+2$. Following the same arguments as in Lemma 5.2, we compute $P_0 = \frac{1}{n}$, $P_1 = -\frac{1}{n(n+2)}$, and arrive at formula (5.5), that is exact for all $u \in H^{2m+2}(B)$. We choose $\bar{u}$, such that

$$\Delta^2 [\bar{u}(\mathbf{x})] = \prod_{i=1}^m (|\mathbf{x}|^2 - \tau_i^2) P_{m-1}(|\mathbf{x}|^2), \quad P_{m-1} \in \pi_{m-1}(\mathbb{R}).$$

Since

$$\Delta^{2m+2} [\bar{u}(\mathbf{x})] = \Delta^{2m} \left[\prod_{i=1}^m (|\mathbf{x}|^2 - \tau_i^2) P_{m-1}(|\mathbf{x}|^2)\right] = 0,$$

it follows that $\bar{u} \in H^{2m+2}(B)$, and (5.5) applied to $\bar{u}$ becomes

$$\int_B (1-|\mathbf{x}|^2)^2 \prod_{i=1}^m (|\mathbf{x}|^2 - \tau_i^2) P_{m-1}(|\mathbf{x}|^2)\, d\mathbf{x} = 0.$$

The latter can be written as

$$\int_0^1 (1-t^2)^2 t^{n-1} \prod_{i=1}^m (t^2 - \tau_i^2) P_{m-1}(t^2)\, dt = 0,$$

which after the change of variables $x = 2t^2 - 1$ is

$$\int_{-1}^1 (1-x)^2 (1+x)^{\frac{n}{2}-1} \prod_{i=1}^m (x+1-2\tau_i^2) P_{m-1}\left(\frac{1+x}{2}\right) dx = 0, \quad P_{m-1} \in \pi_{m-1}(\mathbb{R}).$$

This relation means that the polynomial $\prod_{i=1}^m (x+1-2\tau_i^2) \in \pi_m(\mathbb{R})$, is orthogonal to $\pi_{m-1}(\mathbb{R})$ on $(-1, 1)$ with weight $(1-x)^2(1+x)^{\frac{n}{2}-1}$. This is possible only if $x_i = 2\tau_i^2 - 1$, namely we have that $\tau_i = \sqrt{(1+x_i)/2}$, $i = 1, \ldots, m$, where $\{x_i\}_{i=1}^m$ are the zeroes of the Jacobi polynomial $P_m^{(2,\frac{n}{2}-1)}$. So far, we have uniquely determined the weights P_0, P_1 and the nodes $\{\tau_i\}_{i=1}^m$ of a cubature of type (1.8) with PDP(1.8) $= 2m+2$. Next, we find Q_k from (5.5), applied to the functions $u_k \in H^{2m+2}(B)$, defined by

$$\Delta^2 [u_k(\mathbf{x})] = \omega_k^2(\mathbf{x}), \qquad \omega_k(\mathbf{x}) := \prod_{i=1,\ i\neq k}^m (|\mathbf{x}|^2 - \tau_i^2),$$

and derive that,

$$Q_k = \frac{\int_B (1-|\mathbf{x}|^2)^2 \omega_k^2(\mathbf{x})\, d\mathbf{x}}{8n(n+2)\gamma_n \tau_k^{n-1} \prod_{i=1,\ i\neq k}^m (\tau_k^2 - \tau_i^2)^2}, \quad k = 1, \ldots, m.$$

Table 15
Formula (1.8), $m = 1$, $P_0 = 0.333333$, $P_1 = -0.066667$.

j	1
Q_j	0.001905
τ_j	0.577350

Table 16
Formula (1.8), $m = 2$, $P_0 = 0.333333$, $P_1 = -0.066667$.

j	1	2
Q_j	0.002235	0.000404
τ_j	0.420915	0.769455

Table 17
Formula (1.8), $m = 3$, $P_0 = 0.333333$, $P_1 = -0.066667$.

j	1	2	3
Q_j	0.002109	0.000817	0.000111
τ_j	0.331630	0.627378	0.855402

Table 18
Formula (1.8), $m = 4$, $P_0 = 0.333333$, $P_1 = -0.066667$.

j	1	2	3	4
Q_j	0.001903	0.001029	0.000320	0.000038
τ_j	0.273692	0.527098	0.741454	0.900996

Table 19
Formula (1.8), $m = 5$, $P_0 = 0.333333$, $P_1 = -0.066667$.

j	1	2	3	4	5
Q_j	0.001705	0.001105	0.000502	0.000138	0.000015
τ_j	0.233020	0.453481	0.649506	0.810546	0.928016

Table 20
Formula (1.8), $m = 6$, $P_0 = 0.333333$, $P_1 = -0.066667$.

j	1	2	3	4	5	6
Q_j	0.001533	0.001109	0.000624	0.000255	0.000065	0.000007
τ_j	0.202885	0.397465	0.575777	0.730525	0.855388	0.945326

Table 21
Formula (1.8), $m = 7$, $P_0 = 0.333333$, $P_1 = -0.066667$.

j	1	2	3	4	5	6	7
Q_j	0.001387	0.001079	0.000696	0.000358	0.000136	0.000033	0.000003
τ_j	0.179659	0.353543	0.516066	0.662005	0.786674	0.886076	0.957072

Thus (1.8) is uniquely determined. ■

Remark 5.4. The weights from Theorem 5.3 can be computed explicitly and they are for $j = 1, \ldots, m$,

$$Q_j = \frac{2^{\frac{n}{2}-\frac{9}{2}}(4m+n+6)^2(m+2)}{n(n+2)(m+1)(2m+n+4)^2(2m+n)(2m+n+2)} \cdot \frac{\left(1-x_j\right)\left(1+x_j\right)^{-\frac{n}{2}+\frac{3}{2}}}{\left[P_{m+1}^{(2,n/2-1)}(x_j)\right]^2},$$

where $\{x_j\}_{j=1}^m$ are the zeroes of the Jacobi polynomial $P_m^{(2,n/2-1)}$.

The coefficients and nodes of cubature (1.8) in the three-dimensional case ($n = 3$) are presented in Tables 15–21.

Table 22
Formula (1.4), $m = 1,\ A = 0.133333$.

j	1
B_j	0.600000
C_j	−0.000317
τ_j	0.577350

Table 23
Formula (1.4), $m = 2,\ A = 0.080808$.

j	1	2
B_j	0.412463	0.303093
C_j	−0.000092	−0.000017
τ_j	0.420915	0.769455

Table 24
Formula (1.4), $m = 3,\ A = 0.053333$.

j	1	2	3
B_j	0.324382	0.270757	0.188263
C_j	−0.000032	−0.000013	−0.000002
τ_j	0.331630	0.627378	0.855402

6. Lobatto–Turan cubatures for the ball

Now, we can combine cubatures (1.7) and (1.8), both having PDP $= 2m + 2$, and obtain the Lobatto–Turan cubature of type I and type II.

Theorem 6.1. *For every positive number m, the cubatures of type I, (1.4), and of type II, (1.5), with nodes $\tau_j = \sqrt{(1 + x_j)/2}$, $j = 1, \ldots, m$, where $\{x_j\}_{j=1}^m$ are the zeros of the Jacobi polynomial $P_m^{(2,\frac{n}{2}-1)}$ and weights*

$$A = \frac{E_0P_1 - P_0E_1}{P_1 - E_1}, \qquad B_j = \frac{P_1}{P_1 - E_1}D_j, \qquad C_j = -\frac{E_1}{P_1 - E_1}Q_j, \quad j = 1, \ldots, m,$$

and

$$F = \frac{E_1P_0 - P_1E_0}{P_0 - E_0}, \qquad G_j = \frac{P_0}{P_0 - E_0}D_j, \qquad H_j = -\frac{E_0}{P_0 - E_0}Q_j, \quad j = 1, \ldots, m,$$

respectively, have polyharmonic degree of precision $2m + 2$. Here $E_0,\ E_1,\ \{D_j\}_{j=1}^m$, and $P_0,\ P_1$ and $\{Q_j\}_{j=1}^m$ are the weights from Theorems 4.2 and 5.3.

Proof. We multiply (1.7) (which by Theorem 4.2 exists and is exact for all polyharmonic functions of order $2m + 2$) by P_1 and (1.8) by $-E_1$, add them together and obtain

$$\int_B u(\mathbf{x})\,d\mathbf{x} = \frac{E_0P_1 - P_0E_1}{P_1 - E_1}\int_{S(1)} u(\xi)\,d\sigma(\xi) + \frac{P_1}{P_1 - E_1}\sum_{j=1}^m D_j \int_{S(\tau_j)} u(\xi)\,d\sigma(\xi) - \frac{E_1}{P_1 - E_1}\sum_{j=1}^m Q_j \int_{S(\tau_j)} \Delta^2 u(\xi)\,d\sigma(\xi).$$

Similarly, we derive

$$\int_B u(\mathbf{x})\,d\mathbf{x} = \frac{E_1P_0 - P_1E_0}{P_0 - E_0}\int_{S(1)} \frac{\partial u}{\partial \nu}(\xi)\,d\sigma(\xi) + \frac{P_0}{P_0 - E_0}\sum_{j=1}^m D_j \int_{S(\tau_j)} u(\xi)\,d\sigma(\xi) - \frac{E_0}{P_0 - E_0}\sum_{j=1}^m Q_j \int_{S(\tau_j)} \Delta^2 u(\xi)\,d\sigma(\xi),$$

which is exact for all polyharmonic functions of order $2m + 2$. ■

The coefficients and nodes of cubature (1.4) in the three-dimensional case ($n = 3$) are shown in Tables 22–28. The coefficients and nodes of cubature (1.5) in the three-dimensional case ($n = 3$) are shown in Tables 29–35.

Finally, we demonstrate the PDP of formulas (1.4) and (1.5) in the two dimensional case ($n = 2$). We apply both formulas to the function $u(x, y) = (x^2 + y^2)^{2m+1} \in H^{2m+2}(B)$ and show that they are exact up to the machine error, see Table 36.

Table 25
Formula (1.4), $m = 4$, $A = 0.037559$.

j	1	2	3	4
B_j	0.268732	0.238475	0.190583	0.128869
C_j	−0.000013	−0.000007	−0.000002	−0.000000
τ_j	0.273692	0.527098	0.741454	0.900996

Table 26
Formula (1.4), $m = 5$, $A = 0.027778$.

j	1	2	3	4	5
B_j	0.229640	0.210905	0.180811	0.141005	0.093821
C_j	−0.000006	−0.000004	−0.000002	−0.000001	−0.000000
τ_j	0.233020	0.453481	0.649506	0.810546	0.928016

Table 27
Formula (1.4), $m = 6$, $A = 0.021333$.

j	1	2	3	4	5	6
B_j	0.200521	0.188123	0.168027	0.141059	0.108344	0.071360
C_j	−0.000003	−0.000002	−0.000001	−0.000001	−0.000000	−0.000000
τ_j	0.202885	0.397465	0.575777	0.730525	0.855388	0.945326

Table 28
Formula (1.4), $m = 7$, $A = 0.016878$.

j	1	2	3	4	5	6	7
B_j	0.177954	0.169329	0.155262	0.136208	0.112781	0.085751	0.056095
C_j	−0.000002	−0.000001	−0.000001	−0.000000	−0.000000	−0.000000	−0.000000
τ_j	0.179659	0.353543	0.516066	0.662005	0.786674	0.886076	0.957072

Table 29
Formula (1.5), $m = 1$, $F = 0.044444$.

j	1
G_j	1.000000
H_j	−0.001799
τ_j	0.577350

Table 30
Formula (1.5), $m = 2$, $F = 0.021333$.

j	1	2
G_j	0.544451	0.400082
H_j	−0.000837	−0.000151
τ_j	0.420915	0.769455

Table 31
Formula (1.5), $m = 3$, $F = 0.012698$.

j	1	2	3
G_j	0.386169	0.322329	0.224123
H_j	−0.000440	−0.000171	−0.000023
τ_j	0.331630	0.627378	0.855402

Table 32
Formula (1.5), $m = 4$, $F = 0.008466$.

j	1	2	3	4
G_j	0.302857	0.268758	0.214784	0.145233
H_j	−0.000257	−0.000139	−0.000043	−0.000005
τ_j	0.273692	0.527098	0.741454	0.900996

Table 33
Formula (1.5), $m = 5$, $F = 0.006061$.

j	1	2	3	4	5
G_j	0.250517	0.230079	0.197248	0.153824	0.102350
H_j	−0.000162	−0.000105	−0.000048	−0.000013	−0.000001
τ_j	0.233020	0.453481	0.649506	0.810546	0.928016

Table 34
Formula (1.5), $m = 6$, $F = 0.004558$.

j	1	2	3	4	5	6
G_j	0.214232	0.200986	0.179516	0.150704	0.115752	0.076239
H_j	−0.000108	−0.000078	−0.000044	−0.000018	−0.000005	−0.000000
τ_j	0.202885	0.397465	0.575777	0.730525	0.855388	0.945326

Table 35
Formula (1.5), $m = 7$, $F = 0.003556$.

j	1	2	3	4	5	6	7
G_j	0.187445	0.178360	0.163543	0.143472	0.118797	0.090324	0.059087
H_j	−0.000076	−0.000059	−0.000038	−0.000020	−0.000007	−0.000002	−0.000000
τ_j	0.179659	0.353543	0.516066	0.662005	0.786674	0.886076	0.957072

Table 36
Errors for (1.4) and (1.5) for $u(x, y) = (x^2 + y^2)^{2m+1}$.

m	1	2	3	4	5	6	7
(1.4)	0	$3.5 * 10^{-18}$	$5.2 * 10^{-17}$	$6.9 * 10^{-18}$	$8 * 10^{-17}$	$2.5 * 10^{-16}$	$8 * 10^{-17}$
(1.5)	0	$5.6 * 10^{-17}$	$3.3 * 10^{-16}$	$2.2 * 10^{-16}$	$5.6 * 10^{-16}$	$8.6 * 10^{-16}$	$2.8 * 10^{-16}$

Acknowledgments

The authors would like to thank Professor Dimitar K. Dimitrov for the fruitful discussions on the topic and Matthew Hielsberg for the help in carrying the numerical simulations. This research was supported by Office of Naval Research Contract ONR N00014-11-1-0712, the NSF Grant DMS-1222715, and the Bulgarian National Science Fund Grant DFNI-T01/0001.

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