Participation Constraints in First-Price Auctions*

Xiaoyong Cao† Shao-Chieh Hsueh‡ Guoqiang Tian§ Wei Wang¶

April 11, 2023

Abstract

We study the endogenous participation problem when bidders are characterized by a two-dimensional private information on valuations and participation costs in first-price auctions. Bidders participate whenever their private costs are less than or equal to the expected revenue from participating. We show that there always exists an equilibrium in this general setting with two dimensional types of ex ante heterogeneous bidders. When bidders are ex ante homogeneous, there is a unique symmetric equilibrium, but asymmetric equilibria may also exist. We provide conditions under which the equilibrium is unique (not only among symmetric ones). In the symmetric equilibrium, we show that the equilibrium cutoff of participation costs described above which bidders never participate is lower when the distribution of participation costs is first-order stochastically dominated.

Keywords: Existence and uniqueness; Two-dimensional private information; First-price auctions; Participation costs.

JEL Classification: D42, D62, D82

---

*All authors have no conflicts of interests.

†Department of Economics, University of International Business and Economics, Beijing, China, 100029. Financial supports from the National Natural Science Foundation of China (NSFC-71740016) and the Established Scholar program from University of International Business and Economics (18JQ05) are gratefully acknowledged. Email: yongcx2000@uibe.edu.cn.

‡Institute of Economics, School of Economics and Wang Yanan Institute for Studies in Economics, Xiamen University, Xiamen, Fujian, China, 361005. Email: schsuechieh@gmail.com. Financial support from the Ministry of Education of China (16YJC790116), and from the National Natural Science Foundation of China (NSFC-71403223) are gratefully acknowledged. Email: schsuechieh@gmail.com.

§Department of Economics, Texas A&M University, College Station, TX 77840. (e-mail: tian@econmail.tamu.edu).

¶Corresponding author. School of International Trade and Economics, University of International Business and Economics, 10 Huixin East Street, Beijing 100029, China (e-mail: weiwangccer@gmail.com). The financial support from the National Natural Science Foundation of China (Grant No. 71703017), Beijing Natural Science Foundation (Grant No. 9184031), and the "Fundamental Research Funds for the Central Universities" in UIBE (Grant No. CXTD10-01) are gratefully acknowledged.

1 Introduction

Auctions are an efficient way to allocate scarce resources. However, bidders are often facing participation costs that are usually privately informed. With the existence of participation costs, not all potential bidders will participate in auctions.[1] In practice, usually only a small fraction of potential bidders participate in various auctions.[2] This has some policy implications if the seller wants to eliminate the lower tail of bidders in the distribution of willingness to pay. Bidders with low willingness to pay will stay out of the auction since they are comparing their expected revenue with their private participation costs. The remaining participants take this into account when they are bidding. Participation decisions, along with bidding strategies, should be determined simultaneously in equilibrium.

Auctions with participation costs have been investigated substantially in the literature since the salient work by Green and Laffont (1984), who study participation constraints in the Vickrey auction in a two-dimensional uniform private setting. This literature mostly focuses on second-price auctions, since the bidding behavior in second-price auctions is very straightforward. When a bidder finds it is optimal to participate in a second-price auction, he can not be better than bidding his true value. To characterize the participation equilibrium, the literature usually assume either the participation cost or the value is privately informed while the other is common knowledge (Campbell (1998); Tan and Yilankaya (2006); Cao and Tian (2013)). Some recent papers deal with the case in which both participation costs and value are private information (Gal et al. (2007), Cao et al. (2018)), i.e., they characterize the participation equilibrium in a two-dimensional private setting.

First-price auctions with participation costs are not well-studied, since it is technically difficult to characterize the bidding functions as well as the expected revenue from participation. The difficulty hinges on that, in first-price auctions, bidding strategies are not so explicit compared to those in second-price auctions. Samuelson (1985) studies the symmetric cutoff threshold entrance equilibrium of first-price competitive procurement auctions.[3] Cao and Tian (2010) study the equilibria of first-price auctions when participation costs are common to all bidders. Cao et al. (2019) further study a similar problem in a two-dimensional setting where values are drawn from a binary distribution. This paper extends the existing literature on first price auctions with participation costs to a general

[1]Related terminology includes participation cost, participation fee, entry cost or opportunity costs of participating in the auction. Participating costs differs from entry fee in that entry fee is part of the seller's revenue while participation cost is more general. As we only study the bidder's problem in this paper, we do not distinguish these two terminology.

[2]Hendricks, Pinse and Porter (2003) find only around 25 percent of potential bidders participate in the auctions that took place within 15 years in the united states. Bajari and Hortacsu (2003), Li and Zheng (2009), Athey, Levin and Seira (2011), Li and Zhang (2010, 2014), Krasnokutskaya and Seim (2011), Roberts and Sweeting (2013) for various auctions. See Feng, Lu and Sun (2016) for details.

[3]Another example is Menezes and Monteiro (2000).

two-dimensional private setting.

We present a general two-dimensional private setting on both values and participation costs in a first-price auction and characterize the equilibrium. Bidders can be heterogenous in their types. We focus on the cutoff strategy, assuming that, all bidders participate whenever their private costs are less than or equal to some critical value, which is characterized by the expected payoff from participating, taking account of other bidders' participation decision. To characterize the equilibrium in cutoff strategies we first convert the equilibrium conditions for a profile of cutoff strategies to a system of integral equations. We then use the Schauder-Tychonoff fixed-point theorem to show that there exists a solution to the system of integral equations, which establish the existence of the equilibria. We further investigate the uniqueness of the equilibrium. Particularly, when all bidders are homogeneous in their types, we show there exists a unique symmetric equilibrium, i.e., all bidders use the same cutoff curves and the same bidding functions which is contingent on the number of other participants. By applying the contraction mapping theorem, in a two-bidder economy with general distributions, we show that under some mild restrictions on the joint density of values and participation costs, the equilibrium is unique. Our result shows that when the joint density is uniformly bounded, the equilibrium is unique. Approximately speaking, this happens when the distribution is more dispersed.

We also compare symmetric equilibria with different distributions of participation costs. Consider two symmetric equilibria, A and B, where the distribution of participation costs in equilibrium A first-order stochastically dominates that in equilibrium B. We show that the cutoff function in equilibrium A is higher than those in equilibrium B. Intuitively, if the participation costs are more concentrated in high values, then the expected cost to participate of all bidders is higher. For each bidder i, this implies that other bidders are less likely to participate, leading to a larger expected revenue for bidder i. Hence, bidder i is willing to pay more to participate into the auction.

The main contribution of our paper is as follows. Firstly, this is the first paper to investigate the participation in first price auctions in a two-dimensional private setting, which is significant from both the theoretical and empirical point as first price auctions are used more widely in practice. Secondly, our work is technically more challenging as we have to deal with both the symmetric and even asymmetric first price auction. As such, it is technically more difficult to find the cutoffs since they are determined by the expected revenue from participating in the auction at the thresholds, which in turn depends on the more complicated bidding functions of bidders who submit bids. Last but not the least, we provide a sufficient condition for the uniqueness of the equilibrium.

The most closely related paper is Cao et al. (2018). Our findings show that most results in second-price auctions with private participation costs still hold in first-price auctions with private participation costs. However, our results differ from theirs in at least two important aspects. First, the sufficient condition for the uniqueness of equilibrium of our paper is weaker (more general) than theirs. Second, we can compare symmetric equilibria with different distributions of participation costs if these distributions can be ranked by the first-order stochastic dominance.

The remainder of the paper is organized as follows. We describe the model in Section 2. In Section 3, we characterize the equilibrium cutoff and investigate the existence of the equilibrium. The uniqueness is addressed in Section 4. Concluding remarks are given in Section 5.

2 Model

Consider an independent value environment with one seller and n risk-neutral buyers (bidders). Let $N = \{1, 2, \cdots, n\}$ and $N_{-i} = N/\{i\}$. The seller has an indivisible object which he values at zero. The bidder with the highest bid wins the auction and pays the price equal to his bid. His payoff is equal to the difference between his valuation and the price. The other bidders have zero payoff from submitting a bid. If the highest bid is submitted by more than one bidder, there is a tie which will be broken by a fair lottery. When a bidder submits a bid, he knows who others also submit bids and thus, he can submit a contingent bid based on the identity of his competitors in the auction.

Basically, we are considering a two-stage game. In the first stage, all bidders simultaneously determine whether or not to participate in a first price auction. If a bidder chooses to submit a bid, he pays his participation cost that is not refundable, otherwise the game ends for him. In the second stage, all the bidders who pay the participation costs observe who else also participates in the auction and submit a bid. In order to submit a bid, bidder i must incur a non-refundable participation cost c_i. Bidder i's value for the object, v_i, and his participation cost c_i are independently drawn from the distribution function $K_i(v, c)$, with support $[0, 1] \times [0, 1]$.[4] Let $k_i(v, c) \geq 0$ be the corresponding density function.[5] Assume that $K_i(v, c)$ is continuously differentiable, for all $i \in N$.

The individual action set for any bidder can be characterized as $No \cup [0, 1]$, where "No" denotes not submitting a bid. Bidder i incurs the participation cost c if and only if his action is different from "No". In first-price auctions, given the strategies of all other bidders, as suggested by Lu and Sun (2007)

[4] The support for valuations is normalized to be $[0, 1]$. Bidders with participation costs higher than 1 will not participate in the auction and such bidders are of no practical interest. If the upper bounds of the supports for the participation costs are higher than 1, the above distributions on the participation costs should be interpreted as the truncated distributions of the original distributions on $[0, 1]$.

[5] We will study the special case where v_i and c_i are independently distributed in Sections 4 and 5. When there are atoms in the distribution, $k_i(v_i, c_i)$ can incorporate Dirac delta functions to handle the infinite density.

that for any auction mechanism with participation costs, the participating and nonparticipating types of any bidder are divided by a non-decreasing and equicontinuous shutdown curve. Thus, a bidder submits a bid if and only if his expected payoff is greater than or equal to a cutoff point for costs and does not enter otherwise.

An equilibrium strategy of each bidder i is then determined by the expected payoff of participating in the auction $c_i^*(v)$ when his value is v, together with how to bid in the first-price auction depending on the participation decision of other bidders. In equilibrium, $c_i^*(v)$ depends on the distributions of all bidders' valuations and participation costs. We can interpret $c_i^*(v)$ as the maximal amount bidder i would like to pay to participate in the auction when his value is v. Since participation costs are non-refundable (sunk costs), when a bidder decides to bid, his bidding strategy does not depend on his participation costs. Let $\mathbf{c}^*(v) = \{c_1^*(v), c_2^*(v), \cdots, c_n^*(v)\}$ denote the set of cutoff functions of all bidders and $\mathbf{c}_{-i}^*(v) := \mathbf{c}^*(v) / \{c_i^*(v)\}$.

Before describing bidding strategies, we need to introduce a few notations. Notice that when k other bidders compete with bidder i, there could be different group of bidders and the number of groups is C_{n-1}^k. Denote $O_i^{k_l}$, $i \in N$, $k = \{0, 1, 2, \cdots, n-1\}$, and $l \in \left\{1,, 2, \cdots, C_{n-1}^k\right\}$, as the set of the l-th group with k bidders who also participate in the auction when bidder i participates. Define $O_i^k = \cup_{l \in \left\{1,,2,\cdots,C_{n-1}^k\right\}} O_i^{k_l}$ as the set of all groups with k other participating bidders. Then, define $O_i = \cup_{k \in \{1,,2,\cdots,n-1\}} O_i^k$ as the set of all groups with at least one other participating bidder.

Let $S_i^{k_l} = \left\{c_j^*(v), b_j, K_j(c_j, v_j)\right\}_{j \in O_i^{k_j}}$, $i \in N$, $k = \{1, 2, \cdots, n-1\}$, and $l \in \left\{1,, 2, \cdots, C_{n-1}^k\right\}$, denote the set of the distributions of valuations and participation costs as well as cutoff functions and bidding strategies of all other bidders in the set of the l-th group with k bidders who also participate in the auction when bidder i participates. Define $S_i^k = \cup_{l \in \left\{1,,2,\cdots,C_{n-1}^k\right\}} S_i^{k_l}$ and $S_i = \cup_{k \in \{1,,2,\cdots,n-1\}} S_i^k$ accordingly. Then, the bidder i's strategy when he faces the l-th group of other k bidders can be characterized by:

$$b_i^{k_l}\left(v_i, c_i, S_i^{k_l}\right) = \begin{cases} b_i^{k_l}\left(v_i, S_i^{k_l}\right), & \text{if } c_i \leq c_i^*(v_i) \\ No, & \text{if } c_i > c_i^*(v_i) \end{cases}, \tag{1}$$

where $b_i^{k_l}\left(v_i, S_i^{k_l}\right)$ is a contingent bidding function when bidder i participates in the auction. Assume $b_i^{k_l}\left(v_i, S_i^{k_l}\right)$ is differentiable with respect to v_i for all $i \in N$. Note that, if bidder i enters the auction while the other bidder does not, bidder i bids zero. If the other bidder also participate, the bid of bidder i depends on 1) his own value; 2) the distributions of valuations and participation costs of other bidders; and 3) cutoff functions and bidding strategies of other bidders. In equilibrium, bidder i with value v_i is indifferent between participating and not participating if $c_i = c_i^* (v_i)$.[6] For simplicity, we use $b_i^{k_l}(v_i)$ to denote $b_i^{k_l}\left(v_i, S_i^{k_l}\right)$ hereafter. Define $b_i^k(\cdot) = \left\{b_i^{k_l}(\cdot)\right\}_{l\in\left\{1,,2,\cdots,C_{n-1}^k\right\}}$ and $b_i(\cdot) = \left\{b_i^k(\cdot)\right\}_{k\in\{1,,2,\cdots,n-1\}}$ accordingly. Denote $\mathbf{b}(v) = \{b_1(v), b_2(v), \cdots, b_n(v)\}$ as the set of bidding strategies of all bidders and $\mathbf{b}_{-i}(v) := \mathbf{b}^*(v) / \{b_i(v)\}$. Let $v_i^{k_l}\left(b_i^{k_l}\right)$ denote the corresponding inverse bidding function. Define $v_i^k(\cdot) = \left\{v_i^{k_l}(\cdot)\right\}_{l\in\left\{1,,2,\cdots,C_{n-1}^k\right\}}$ and $v_i(\cdot) = \left\{v_i^k(\cdot)\right\}_{k\in\{1,,2,\cdots,n-1\}}$ accordingly.

For the game described above, each bidder's action is to choose a cutoff for each value, i.e., a cutoff function, and decide how to bid when he participates. Thus, a (Bayesian-Nash) equilibrium of the sealed-bid first-price mechanism with participation costs is composed of bidders' cutoff strategies and participants' bidding strategies. Formally, we define the strategy profiles as follows.

Definition 1. A strategy profile $(\mathbf{c}^*(v), \mathbf{b}(v))$ is a (Bayesian-Nash) equilibrium of the first-price auction with private participation costs if action $(c_i^*(v), b_i(v))$ maximizes the expected revenue of bidder i, given the strategies of other bidders $\left(\mathbf{c}_{-i}^*(v), \mathbf{b}_{-i}(v)\right)$.

Note that, once the cutoff functions $\mathbf{c}^*(v)$ are determined, the game is reduced to the standard first-price auction and the optimal bidding functions for participating bidders are uniquely determined (see Maskin and Riley (2003)). As such, an equilibrium is fully characterized by the set of cutoffs $\mathbf{c}^*(v)$. Then, all the results in the paper should be interpreted in terms of cutoff functions. In general, different bidders have different cutoff functions since the distributions $K_i(v, c)$ is bidder-specific. However, when the distributions are the same across all bidders, i.e., $K_i(v, c) = K(v, c)$ for all $i \in N$, we can define a symmetric equilibrium.

Definition 2. If $K_i(v, c) = K(v, c)$ for all $i \in N$, an equilibrium $(\mathbf{c}^*(v), \mathbf{b}(v))$ of the first-price auction with participation costs is symmetric equilibrium if the bidders have the same cutoff function, i.e., $c_i^*(v) = c^*(v)$ for all $i \in N$.

When bidders are homogenous and use the same cutoff function, all bidders will share the same bidding function, which is contingent on the number of competitors in the auction.

Remark 1. $c_i^*(v_i) = 1$ implies bidder i with valuation v_i always participates and $c_i^*(v_i) = 0$ implies bidder i with valuation v_i never participates.

[6] The description of the equilibria can be slightly different under different informational structures on $K_i(v_i, c_i)$. For example, when v_i is private information and c_i is exogenously fixed for all bidders, $K_i(v_i, c_i) = K_i(v_i)$ (See Cao and Tian, 2010), the equilibrium is described by a valuation cutoff v_i^* for bidder i such that bidder i submits a bid whenever $v_i \geq v_i^*$.

3 Equilibrium

Suppose, for the time being, there exists an equilibrium where bidders use the cutoff functions $\mathbf{c}^* (v)$ as their participation strategy. Bidder i with value v will participate in the auction and submit a bid that is contingent on information of other participating bidders S_i if and only if $c_i \leq c_i^* (v_i)$. Bidder i with value v_i can win the object being auctioned only when the following two cases happen. Firstly, there is no other bidder participating. This happens with probability $\prod_{j\in N_{-i}} \int_0^1 \int_{c_j^*(\tau)}^1 k_j (c, \tau) \, dcd\tau$. Bidder i just bids zero in this case.

In the second case, there are at least one other bidder participating the auction yet all other bidders bid less than what bidder i bids. As we describe above, bidder i could face many different group of opponents. Consider the case where the l-th group of k other bidders participate yet all these k bidders bid less than $b_i^{k_l}$. In this case, bidder i with value v bids b_i to maximize his expected revenue:

$$\max_{b_i} (v - b_i) \, P\left(b_i; O_i^{k_l}\right) Q\left(N_{-i}/O_i^{k_l}\right), \tag{2}$$

where

$$P\left(b_i; O_i^{k_l}\right) := \prod_{j\in O_i^{k_l}} \int_0^{v_j^{k_l\prime}(b_i)} \int_0^{c_j^*(\tau)} k_j (c, \tau) \, dcd\tau, \; Q\left(N_{-i}/O_i^{k_l}\right) := \prod_{m\in N_{-i}/O_i^{k_l}} \int_0^1 \int_{c_m^*(\tau)}^1 k_m (c, \tau) \, dcd\tau.$$

The first part is the revenue conditional on winning. The second part is the probability of facing the l-th group of k other bidders who all bid less than b. The last part is the probability of all other bidders do not participate. To ensure these probabilities are well-defined, we assume

$$\prod_{j\in O_i^{k_l}} \int_0^{v_j^{k_l\prime}(b_i)} \int_0^{c_j^*(\tau)} k_j (c, \tau) \, dcd\tau = 1, \quad \prod_{m\in N_{-i}/O_i^{k_l}} \int_0^1 \int_{c_m^*(\tau)}^1 k_m (c, \tau) \, dcd\tau = 1,$$

whenever $O_i^{k_l}$ and $N_{-i}/O_i^{k_l}$ are empty sets, respectively, for all $i \in N, \; k = \{0, 1, 2, \cdots, n-1\}$, and $l \in \left\{1, , 2, \cdots, C_{n-1}^k\right\}$.

Notice that the bidding strategy $b_i^{k_l} (v)$ solves this problem. Maskin and Riley (2003) show that there exists a unique bidding equilibrium for the above first-price auction. Denote the maximized expected revenue of bidder i when he competes with the l-th group of k other bidders by:

$$R_i^{k_l} (v) = \left(v - b_i^{k_l} (v)\right) P\left(b_i^{k_l}; O_i^{k_l}\right) Q\left(N_{-i}/O_i^{k_l}\right). \tag{3}$$

Hence, the total maximized expected revenue of bidder i with value v when there is at least one opponent is given by:

$$R_i(v) = \sum_{k=1}^{n-1} \sum_{l=1}^{C_{n-1}^k} R_i^{k_l}(v). \tag{4}$$

Therefore, in equilibrium, the following condition must hold:

$$c_i^*(v) = v\left(\prod_{j\in N_{-i}} \int_0^1 \int_{c_j^*(\tau)}^1 k_j(c,\tau)\, dcd\tau\right) + R_i(v), \tag{5}$$

for all $i \in N$. For the sake of simplicity, we define $\Phi_i(v)$ as follows:

$$\Phi_i(s) := \prod_{j\in N_{-i}} \int_0^1 \int_{c_j^*(\tau)}^1 k_j(c,\tau)\, dcd\tau + \sum_{k=1}^{n-1} \sum_{l=1}^{C_{n-1}^k} P\left(b_i^{k_l}(s); O_i^{k_l}\right) Q\left(N_{-i}/O_i^{k_l}\right).$$

Notice that $\Phi_i(v)$ is the probability of winning for bidder i with value v. Notice that O_i^0 is empty, we have $N_{-i} = N_{-i}/O_i^0$ and, for all $i \in N$,

$$\left(\prod_{j\in O_i^0} \int_0^{v_j^{k_l\prime}\left(b_i^{k_l}(v)\right)} \int_0^{c_j^*(\tau)} k_j(c,\tau)\, dcd\tau\right)\left(\prod_{j\in N_{-i}/O_i^{k_l}} \int_0^1 \int_{c_j^*(\tau)}^1 k_j(c,\tau)\, dcd\tau\right) = \prod_{j\in N_{-i}} \int_0^1 \int_{c_j^*(\tau)}^1 k_j(c,\tau)\, dcd\tau.$$

Hence, we can rewrite $\Phi_i(v)$ as follows:

$$\Phi_i(v) = \sum_{k=0}^{n-1} \sum_{l=1}^{C_{n-1}^k} P\left(b_i^{k_l}(v); O_i^{k_l}\right) Q\left(N_{-i}/O_i^{k_l}\right),$$

for all $i \in N$.

Next, let $\bar{P}\left(b_i^{k_l}(v); O_i^{k_l}\right)$ denote the probability of losing for bidder i in $O_i^{k_l}$ when his bid is $b_i^{k_l}(v)$, for $i \in N$, $k = \{1, 2, \cdots, n-1\}$, and $l \in \left\{1, , 2, \cdots, C_{n-1}^k\right\}$. For each $O_i^{k_l}$, rank bidders by their valuations in ascending order. In particular, if the valuation of a bidder in $O_i^{k_l}$ is the p-th largest, the index of this bidder is $p \in \{1, 2, \cdots, k\}$. Define the following two auxiliary probabilities:

$$\underline{Q}_p^{k_l} = \begin{cases} \prod_{j=1}^{p-1} \int_0^{v_j^{k_l\prime}\left(b_i^{k_l}(v)\right)} \int_0^{c_j^*(\tau)} k_j(c,\tau)\, dcd\tau, & \text{if } 2 \le p \le k \\ 1, & \text{if } p = 1 \end{cases},$$

$$\bar{Q}_p^{k_l} = \begin{cases} \prod_{j=p+1}^{k} \int_0^1 \int_0^{c_j^*(\tau)} k_j\left(c,\tau\right) dcd\tau, & \text{if } 1 \leq p \leq k-1 \\ 1, & \text{if } p = k \end{cases}.$$

By the definition of $\bar{P}\left(b_i^{k_l}\left(v\right); O_i^{k_l}\right)$, we have:

$$\bar{P}\left(b_i^{k_l}\left(v\right); O_i^{k_l}\right) := \sum_{p=1}^{k} \left(\left(\left(\int_{v_p^{k_l\prime}\left(b_i^{k_l}(v)\right)}^{1} \int_0^{c_p^*(\tau)} k_p\left(c,\tau\right) dcd\tau \right) \underline{Q}_p^{k_l} \bar{Q}_p^{k_l} \right).$$

Now, since $\Phi_i\left(\cdot\right)$ is the probability of winning for bidder i, we have:

$$\Phi_i\left(v\right) = 1 - \sum_{k=0}^{n-1} \sum_{l=1}^{C_{n-1}^k} \left(\bar{P}\left(b_i^{k_l}\left(v\right); O_i^{k_l}\right) Q\left(N_{-i}/O_i^{k_l}\right)\right). \tag{6}$$

Proposition 1 solves the expected revenue maximization problem and derives the equilibrium condition that the cutoff function has to satisfy.

Proposition 1. *Suppose there exists an equilibrium cutoff $\mathbf{c}^*\left(v\right)$ and all bidding functions are differentiable. Then, $\mathbf{c}^*\left(v\right)$ is characterized by:*

$$c_i^*\left(v\right) = \int_0^v \Phi_i\left(s\right) ds, \tag{7}$$

with $c_i^*\left(0\right) = 0$ for all $i \in N$. $c_i^*\left(v\right)$ is differentiable with

$$c_i^{*\prime}\left(v\right) = \Phi_i\left(v\right), \tag{8}$$

for all $i \in N$. $c_i^{*\prime}\left(v\right)$ is also differentiable.

Proof. See the Appendix. □

Notice that the first part of $c_i^{*\prime}\left(v\right)$ is the probability that none of the other $n-1$ bidders participates and the second part is the probability that at least one of the other $n-1$ participating bidders who all bid less than b_i. Hence, $c_i^{*\prime}\left(v\right)$, as the sum of the two parts, is the probability of winning for bidder i. We illustrate the above analysis in the three-bidder example below.

Remark 2. $v_j\left(b_i\left(v\right)\right)$ gives the value of bidder j such that he bids exactly the same as bidder i with value v. When all bidders use the same bidding function, $v_j\left(b_i\left(v\right)\right) = v$. This happens in second price auctions or in a symmetric equilibrium in first price auctions.

Example 1. Let $n=3$. We only characterize the cutoff function of bidder 1 and the case for other two bidders is similar. Notice that $O_1^{1_1}=\{2\}$ and $O_1^{1_2}=\{3\}$. Hence, $O_1^1=\{\{2\},\{3\}\}$. In addition, $O_1^2=\{2,3\}$. Therefore, the expected revenue maximization problem of bidder 1 with value v is:

$$
\begin{aligned}
c_1^*(v) = {} & v\left(\int_0^1\int_{c_2^*(\tau)}^1 k_2(c,\tau)\,dcd\tau\right)\left(\int_0^1\int_{c_3^*(\tau)}^1 k_3(c,\tau)\,dcd\tau\right)\\
& +\max_{b_1^{1_1}}\left(v-b_1^{1_1}\right)\left(\int_0^{v_2^1\left(b_1^{1_1}\right)}\int_0^{c_2^*(\tau)} k_2(c,\tau)\,dcd\tau\right)\left(\int_0^1\int_{c_3^*(\tau)}^1 k_3(c,\tau)\,dcd\tau\right)\\
& +\max_{b_1^{1_2}}\left(v-b_1^{1_2}\right)\left(\int_0^{v_3^1\left(b_1^{1_2}\right)}\int_0^{c_3^*(\tau)} k_3(c,\tau)\,dcd\tau\right)\left(\int_0^1\int_{c_2^*(\tau)}^1 k_2(c,\tau)\,dcd\tau\right)\\
& +\max_{b_1^2}\left(v-b_1^2\right)\left(\int_0^{v_2^2\left(b_1^2\right)}\int_0^{c_2^*(\tau)} k_2(c,\tau)\,dcd\tau\right)\left(\int_0^{v_3^2\left(b_1^2\right)}\int_0^{c_3^*(\tau)} k_3(c,\tau)\,dcd\tau\right).
\end{aligned}
$$

The first line is the payoff when all other bidders do not participate. The second and third line are the payoff when bidder 3 and 2 does not participate and bidder 2 and 3 participates but bids less than what bidder 1 does. The last line is the payoff when both other bidders participate but bid less. By the envelope theorem, we have

$$
\begin{aligned}
c_1^{*\prime}(v) = {} & \left(\int_0^1\int_{c_2^*(\tau)}^1 k_2(c,\tau)\,dcd\tau\right)\left(\int_0^1\int_{c_3^*(\tau)}^1 k_3(c,\tau)\,dcd\tau\right)\\
& +\left(\int_0^{v_2^1\left(b_1^{1_1}(v)\right)}\int_0^{c_2^*(\tau)} k_2(c,\tau)\,dcd\tau\right)\left(\int_0^1\int_{c_3^*(\tau)}^1 k_3(c,\tau)\,dcd\tau\right)\\
& +\left(\int_0^{v_3^1\left(b_1^{1_2}(v)\right)}\int_0^{c_3^*(\tau)} k_3(c,\tau)\,dcd\tau\right)\left(\int_0^1\int_{c_2^*(\tau)}^1 k_2(c,\tau)\,dcd\tau\right)\\
& +\left(\int_0^{v_2^2\left(b_1^2(v)\right)}\int_0^{c_2^*(\tau)} k_2(c,\tau)\,dcd\tau\right)\left(\int_0^{v_3^2\left(b_1^2(v)\right)}\int_0^{c_3^*(\tau)} k_3(c,\tau)\,dcd\tau\right).
\end{aligned}
$$

Before establishing the existence and uniqueness of the equilibrium, we first characterize some useful properties of the equilibrium cutoff function $c^*(v)$. Proposition 2 summarizes these properties that will be used in the proof of Theorem 1.

Proposition 2. *Suppose there exists an equilibrium cutoff $\mathbf{c}^*(v)$. It has the following properties: for $i\in N$,*

(i) $c_i^{*\prime}(v)\geq 0$ *and* $c_i^{*\prime\prime}(v)\geq 0$*;*

(ii) $0\leq c_i^*(v)\leq v$*;*

(iii) $c_i^{*\prime}(1)=1$.

Proof. See the Appendix. □

The first part of Proposition 2 states that the expected payoff from participating of a bidder is increasing and convex in his value v. The second part shows that a bidder will not be willing to pay more than his value to participate in the auction. The last part finds that the marginal willingness to pay to participate in the auction of the bidder with value $v = 1$ is also 1. The intuition is that when his value for the object is 1, he will almost surely win the object, and the marginal willingness to pay is equal to his value which is 1.

We establish our main results, which are the existence and uniqueness of the symmetric equilibrium, in Theorem 1.

Theorem 1. *There always exists an equilibrium cutoff function* $\mathbf{c}^*(v)$ *such that all bidders participate whenever* $c_i \leq c_i^*(v)$ *and equation (7) holds for all* $i \in N$ *simultaneously.*

Proof. See the Appendix. □

Remark 3. In the proof, we construct a mapping of $\mathbf{c}^*(v)$ from a space to itself. In the Appendix, we show that this space is a compact convex nonempty subset of a locally convex topological space and the mapping is continuous. Then we establish the existence of equilibrium using the Schauder-Tychonoff fixed-point theorem, which states that any continuous mapping from a nonempty compact convex subset of a locally convex topological space to itself has a fixed point

4 Uniqueness

In this section, we investigate the uniqueness of equilibrium in two interesting yet important settings. On the one hand, we establish the uniqueness of equilibrium in the case where all bidders are ex ante homogeneous. For any two symmetric equilibria where valuations and participation costs are independently distributed, we show that the equilibrium cutoff is lower when the distribution of participation costs is first-order stochastically dominated. On the other hand, we show there is a unique equilibrium in the case where the distribution functions are heterogeneous across bidders.

4.1 Symmetric Equilibrium

We first establish uniqueness of symmetric equilibrium. Assume all bidders are homogenous with a common joint distribution of valuations and participation costs. $K_i(\cdot,\cdot) = K(\cdot,\cdot)$ and $k_i(\cdot,\cdot) = k(\cdot,\cdot)$ for all $i \in N$. We focus on the symmetric equilibrium in which all bidders share the same cutoff function $c_i^*(v) = c^*(v)$ and the same bidding function $b_i(v) = b(v)$, for all $i \in N$. In this case, once a bidder participates, his bid depends only on the number of other participating bidders. Let $b^k(v)$ be the bidding function of a bidder with value v when there are k other participants. Then, the cutoff function satisfies:

$$
\begin{aligned}
c^*(v) = & \, v\left[\int_0^1 \int_{c^*(\tau)}^1 k(c,\tau)\, dcd\tau\right]^{n-1} \\
& + \sum_{k=1}^{n-1} C_{n-1}^k \left(\max_{b^k} \left(v - b^k\right) \left(\int_0^{v^k\left(b^k\right)} \int_0^{c^*(\tau)} k(c,\tau)\, dcd\tau \right)^k \left(\int_0^1 \int_{c_j^*(\tau)}^1 k(c,\tau)\, dcd\tau \right)^{n-1-k} \right).
\end{aligned}
$$

Proposition 3 characterizes the cutoff $c^*(v)$ in an symmetric equilibrium.

Proposition 3. *When all bidders are homogenous, the symmetric equilibrium is characterized by*

$$
c^*(v) = \int_0^v \left[1 - \int_t^1 \int_0^{c^*(\tau)} k(c,\tau)\, dcd\tau\right]^{n-1} dt, \tag{9}
$$

with $c^*(v) \in [0, v]$ *and* $c^*(0) = 0$. $c^*(v)$ *is differentiable with*

$$
c^{*\prime}(v) = \left(1 - \int_v^1 \int_0^{c^*(\tau)} k(c,\tau)\, dcd\tau\right)^{n-1} \geq 0,
$$

and $c^{*\prime}(1) = 1$. $c^*(v)$ *is also differentiable with* $c^{*\prime\prime}(v) \geq 0$.

Proof. See the Appendix. □

We illustrate the symmetric equilibrium in the three-bidder example below.

Example 2. Let all bidders are homogenous: $K_i(\cdot,\cdot) = K(\cdot,\cdot)$. The expected revenue maximization problem of any bidder with value v is:

$$
\begin{aligned}
c^*(v) = & \, v\left(\int_0^1 \int_{c^*(\tau)}^1 k(c,\tau)\, dcd\tau\right)^2 \\
& + 2\max_{b^1} \left(v - b^1\right) \left(\int_0^{v^1\left(b^1\right)} \int_0^{c^*(\tau)} k(c,\tau)\, dcd\tau \right) \left(\int_0^1 \int_{c^*(\tau)}^1 k(c,\tau)\, dcd\tau \right) \\
& + \max_{b^2} \left(v - b_1^2\right) \left(\int_0^{v^2\left(b^2\right)} \int_0^{c^*(\tau)} k(c,\tau)\, dcd\tau \right)^2.
\end{aligned}
$$

The first line is the payoff when all other bidders do not participate. The second line is the payoff when only one other bidder participates but bids less than what bidder 1 does. The last line is the payoff when both other bidders participate but bid less. By the envelope theorem, we have

$$
\begin{aligned}
c^{*\prime}(v) = & \left(\int_0^1 \int_{c^*(\tau)}^1 k(c,\tau)\, dcd\tau\right)^2 \\
& + 2\left(\int_0^v \int_0^{c^*(\tau)} k(c,\tau)\, dcd\tau\right)\left(\int_0^1 \int_{c^*(\tau)}^1 k(c,\tau)\, dcd\tau\right) \\
& + \left(\int_0^v \int_0^{c^*(\tau)} k(c,\tau)\, dcd\tau\right)^2 \\
& = \left(1 - \int_v^1 \int_0^{c^*(\tau)} k(c,\tau)\, dcd\tau\right)^2 .
\end{aligned}
$$

We establish the uniqueness of symmetric equilibrium in Theorem 2.

Theorem 2. ***(Uniqueness of Symmetric Equilibrium)*** *Suppose that all bidders have the same distribution function $K(\cdot,\cdot)$. There is a unique symmetric equilibrium where all bidders use the identical cutoff strategy $c^*(\cdot)$.*

Proof. See the Appendix. □

Remark 4. In the literature, Cao and Tian (2010) and Cao et al. (2019) are the very few papers that also establish the existence and uniqueness of equilibrium in first-price auctions with participation costs. However, Cao and Tian (2010) assume that participation costs are common knowledge while the private value follows a binary distribution in Cao et al. (2019). Thus, to some extent, these authors investigate special cases of this paper.

Next, we investigate how the unique symmetric equilibrium cutoff function responds to changes of the distribution of participation costs. Note that Proposition 3 implies that the cutoff function of a symmetric equilibrium is determined in equation (9) solely by the joint distributions of valuations and participation costs. It is interesting to compare equilibrium cutoff functions that corresponds to different distributions of participation costs. To this end, we assume valuations and participation costs are independently distributed, i.e. $K(v,c) = F(v)\, G(c)$ and $k(v,c) = f(v)\, g(c)$with $f(v) = F'(v)$ and $g(c) = G'(c)$. We compare equilibrium cutoff functions of symmetric equilibria, where all bidders share an equilibrium cutoff function $c^*(\cdot)$.

More specifically, consider two sets of distributions A and B, where $F_A(v) = F_B(v)$ and $G_A(c)$ first-order stochastically dominates $G_B(c)$, i.e. $G_A < G_B$.[7] We rewrite equation (9) for $q \in \{A, B\}$ as

[7] Subscripts A and B are used to indicate variables associated with different distributions.

follows:

$$c_q^*(v) = \int_0^v \left[1 - \int_t^1 f_q(\tau)\, G_q\left(c_q^*(\tau)\right) d\tau\right]^{n-1} dt.$$

Theorem 1 and 2 ensure that there is a unique equilibrium for distribution $q \in \{A, B\}$. Proposition 4 presents the comparative statics of the symmetric equilibrium.

Proposition 4. *$c_A^*(v) > c_B^*(v)$ for all $v \in (0, 1]$ if G_A first-order stochastically dominates G_B, where $c_q^*(v)$ is the equilibrium cutoff function for distribution $q \in \{A, B\}$.*

Proof. See the Appendix . □

Remark 5. Intuitively, if the participation costs are more concentrated in high values, then the expected cost to participate of all bidders is higher. For each bidder i, this implies that other bidders are less likely to participate. Hence, the expected revenue for bidder i is larger. This in turn means bidder i is willing to pay more to participate into the auction. Notice also that even though the equilibrium cutoff is higher, the probability of participation for each bidder declines.

4.2 Uniqueness of Equilibrium

Uniqueness of the symmetric equilibrium does not imply the uniqueness of equilibrium. Even when bidders are homogenous, they may still participate in the auction asymmetrically, i.e., they use different cutoff curves. In the next section, under a general distribution, we provide a sufficient condition for the uniqueness of equilibrium.

In this subsection, we consider the case where there are two bidders. In this case, equation (7) and (8) are reduced to:

$$c_i^*(v) = \int_0^v \left[1 - \int_{v_j(b_i(s))}^1 \int_0^{c_j^*(\tau)} k_j(c, \tau)\, dc d\tau\right] ds, \tag{10}$$

$$c_i^{*\prime}(v) = 1 - \int_{v_j(b_i(v))}^1 \int_0^{c_j^*(\tau)} k_j(c, \tau)\, dc d\tau,$$

where $i, j \in \{1, 2\}$ and $i \neq j$. To show the uniqueness of asymmetric equilibrium, we rely on the contraction mapping theorem. Proposition 5 presents our result.

Proposition 5. ***(Uniqueness of Equilibrium)*** *Suppose $n = 2$. If, for $i, j \in \{1, 2\}$ and $j \neq i$, (i) $K_i(c_i, v_i)$ is differentiable; and (ii) $sup_{c,v\in[0,1]} k_i(c, v) < \frac{1}{\int_0^1 [1 - v_i(b_j(v))] dv}$, then there exists a unique equilibrium.*

Proof. See the Appendix. $\square$

Corollary 1. *Suppose $n = 2$. If, for $i, j \in \{1, 2\}$ and $j \neq i$, (i) $K_i(c_i, v_i)$ is differentiable; and (ii) the valuations and participation costs are independently distributed, i.e. $K_i(v_i, c_i) = F_i(v_i) G_i(c_i)$; and (iii) $sup_{c\in[0,1]} g_i(c) < \frac{1}{\int_0^1 [1-F_i(v_i(b_j(v)))]dv}$, then there exists a unique equilibrium. Condition (iii) holds if condition (ii) in Proposition 5 is satisfied.*

Proof. See the Appendix. $\square$

Remark 6. A few remarks on the uniqueness of equilibrium for the two-bidder setting are in order.

(1) The sufficient conditions in Proposition 5 can be easily satisfied. For instance, when participation costs are jointly uniformly distributed on $[0, 1] \times [0, 1]$ for both bidders, the supremum of the density for participation costs is 1, i.e., $\sup_{c,v\in[0,1]} k_i(c, v) = 1$, and clearly $\int_0^1 [1 - v_i(b_j(v))] dv < 1$ for any $K_i(\cdot, \cdot)$ on $[0, 1] \times [0, 1]$. Thus, in this case, the equilibrium is unique regardless of the distributions of valuations and participation costs.

(2) It can be concluded that in the two-bidder economy, when the bidders are homogeneous and when the sufficient conditions in Proposition 5 are satisfied, the symmetric equilibrium is the unique equilibrium.

(3) In the proof of the uniqueness of equilibrium, we make use of the mean value theorem before using the contraction mapping theorem. For $n \geq 3$, there are more product terms in the expected revenue of each bidder, which makes the application of the mean value theorem less tractable.

(4) Approximately speaking, the sufficient condition in Proposition 5 holds when c_i is more dispersed or v_i is more concentrated on low values. The latter is roughly consistent with Cao and Tian (2010), which states that the equilibrium is unique when c_i and v_i are independently distributed and the distribution function of v_i is concave.

5 Concluding Remarks

We study the existence and uniqueness of equilibrium in first-price auctions when bidders' values and participation costs are both private information. We show that under general distribution functions, there always exists an equilibrium in which each bidder uses a cutoff strategy. When bidders are ex ante homogeneous, there is a unique symmetric equilibrium. When there are two heterogeneous bidders, we provide a mild sufficient condition for the uniqueness of the equilibrium. Future research could focus on identifying sufficient conditions that are easier to verify to guarantee uniqueness of equilibrium in general environments. Beside that, this paper only focus on the equilibrium analysis of the bidders. The next step would be to study the implications of the private participation costs on the seller's expected revenue as well as the social welfare. We would leave it for a future work.

References

[1] Athey, S., S., J. Levin, and E. Seira: Comparing open and sealed bid auctions: evidence from timber auctions, *Quarterly Journal of Economics*, 126(1), (2011), 207-257.

[2] Bajari, P. and A. Hortacsu: The winner's curse, reserve prices and endogenous entry: empirical insights from eBay auctions, *RAND Journal of Economics*, 34(2), (2003), 329-355.

[3] Burton, T.A.: *Stability and Periodic Solutions of Ordinary and Functional Differential Equations*. Dover Publications Inc, Mineola, New York (2005).

[4] Campbell, C. M.: Coordination in auctions with entry. *Journal of Economic Theory*, 82, (1998), 425-450.

[5] Cao, X., Tan, G., Tian, G. and Yilankaya, O.: Equilibria in second-price auctions with private participation costs. *Economic Theory*, 65, (2018), 231-249.

[6] Cao, X., Hsueh S., Kung, C. and Yang, Y.: Equilibria in first-price auctions with participation costs. *working paper*, (2019).

[7] Cao, X. and Tian, G.: Second-price auctions with different participation costs. *Journal of Economics and Management Strategy*, 22(1), (2013), 184-205.

[8] Cao, X. and Tian, G.: Equilibria in first-price auctions with participation costs. *Games and Economic Behavior*, 2, (2010), 258-273.

[9] Gal, S., Landsberger, M., and Nemirovski, A.: Participation in auctions. *Games and Economic Behavior*, 60, (2007), 75-103.

[10] Green, J. and Laffont, J.J.: Participation constraints in the Vickrey auction. *Economics Letters*, 16, (1984), 31-36.

[11] Hendricks, K., Pinkse, J. and Porter, R.: Empirical implications of equilibrium bidding in first-price, symmetric, common value auctions. *Review of Economic Studies*, 70(1), (2003), 115-145.

[12] Li, T. and B. Zhang: Testing for Affiliation in first-price auctions using entry behavior. *International Economic Review*, 51(3), (2010), 837-850.

[13] Li, T. and B. Zhang: Affiliation and entry in first-price auctions with heterogeneous bidders: an analysis of merger effects, *American Economic Journal: Microeconomics*, 7(2), (2015), 188-214.

[14] Li, T. and X. Zheng: Entry and competition effects in first-price auctions: theory and evidence from procurement auctions, *Review of Economic Studies*, 76(4), (2009), 1397-1429.

[15] Lu, J. and Sun, Y.: Efficient auctions with private participation costs. working paper, (2007).

[16] Roberts, J. and A. Sweeting: When should sellers use auctions? *American Economic Review*, 103(5), (2013), 1830-61.

[17] Samuelson, W.: Competitive bidding with entry costs, *Economics Letters*, 17, (1985), 53-57.

[18] Tan, G. and Yilankaya, O.: Equilibria in second-price auctions with participation costs. *Journal of Economic Theory*, 130, (2006), 205-219.

Appendix

Proof of Proposition 1

First, notice that it follows from the inverse function theorem that all inverse bidding functions are also differentiable. Hence, $\Phi_i(v)$ and thus $R_i(v)$ are differentiable for all $i \in N$. It follows that $c_i^*(v)$ is differentiable for all $i \in N$. Then, we can apply the envelope theorem in equation (5). Differentiating equation (5) derives:

$$c_i^{*\prime}(v) = \Phi_i(v),$$

for all $i \in N$. Since $\Phi_i(v)$ is differentiable, $c_i^{*\prime}(v)$ is also differentiable. Last, integrating equation(8) from 0 to 1 and noticing $c_i^*(0) = 0$ for all $i \in N$, we derive equation (7). To see why $c_i^*(0) = 0$, notice that since $0 \leq b_i^{k_l}(v) \leq 0$, we have $b_i^{k_l}(0) = 0$, for all $i \in N$, $k = \{1, 2, \cdots, n-1\}$, and $l \in \left\{1, , 2, \cdots, C_{n-1}^k\right\}$. It follows that $\Phi_i(0) = 0$, for all $i \in N$. Hence, evaluating equation (5) at $v = 0$ yields $c_i^*(0) = 0$, for all $i \in N$. This completes the proof.

Proof of Proposition 2

First, (i) equation (8) implies $c_i^{*\prime}(v) \geq 0$ since $K_i(\cdot, \cdot)$ is a distribution function defined on $[0,1] \times [0,1]$. Since $c_i^{*\prime}(v)$ is differentiable for all $i \in N$, it is clear that equation (8) implies $c^{*\prime}(v)$ is increasing:

$$c_i^{*\prime\prime}(v) = \Phi_i'(v) \geq 0,$$

for all $i \in N$. Second, equation (7) implies $c_i^*(v) \geq 0$. Notice that $\Phi_i(\cdot) \in [0,1]$ since it is the probability of winning for bidder i. Equation (7) implies:

$$c_i^*(v) = \int_0^v \Phi_i(s)\, ds \leq \int_0^v ds = v.$$

Last, notice that:

$$c_i^{*\prime}(v) = \Phi_i(v) = 1 - \sum_{k=0}^{n-1} \sum_{l=1}^{C_{n-1}^k} \left(\bar{P}\left(b_i^{k_l}(v); O_i^{k_l}\right) Q\left(N_{-i}/O_i^{k_l}\right)\right),$$

where

$$\bar{P}\left(b_i^{k_l}(v); O_i^{k_l}\right) := \sum_{p=1}^{k} \left(\left(\int_{v_p^{k_l\prime}\left(b_i^{k_l}(v)\right)}^{1} \int_0^{c_p^*(\tau)} k_p(c,\tau)\, dc d\tau\right) \underline{Q}_p^{k_l} \bar{Q}_p^{k_l}\right).$$

Notice that Maskin and Riley (2003) show that if the upper endpoint of the support of the valuation distributions is the same for all bidders in first-price auctions, then the upper endpoints of the supports of all bidders' equilibrium bid distributions are the same. Since v_i have the same upper endpoint of their distributions, we have $b_i(1) = b^\star$ and $v_j(b_i(1)) = 1$, where $j \neq i$ and $b^\star$ is a constant. Evaluating $c_i^{*\prime}(v)$ at $v = 1$ derives $c_i^{*\prime}(1) = 1$ since $v_j^{k_l\prime}\left(b_i^{k_l}(1)\right) = 1$ for all $i \in N$. This completes the proof.

Proof of Theorem 1

We establish the existence by appying the Schauder-Tychonoff fixed-point theorem, which states that any continuous mapping from a compact convex non-empty subset of a locally convex topological space to itself has a fixed point (Burton (2005)). For all $i \in N$, define a mapping $h_i : [0,1] \to [0,1]$ as follows:

$$h_i(t_i, \mathbf{c}^*(\cdot)) := \Phi(t_i; \mathbf{c}^*(\cdot)).$$

Since $K(\cdot,\cdot)$ is integrable over both arguments and bidding functions are differentiable (hence continuous), $h_i(t_i, \mathbf{c}^*(\cdot))$ is a continuous mapping from $[0,1] \times [0,1]^n \to [0,1]$, for all $i \in N$. Define $H(t, \mathbf{c}^*(\cdot)) : [0,1]^n \times [0,1]^n \to [0,1]^n$ as follows:

$$H(t, \mathbf{c}^*(\cdot)) = (h_1(t_1, \mathbf{c}^*(\cdot)), h_2(t_2, \mathbf{c}^*(\cdot)), \cdots, h_n(t_n, \mathbf{c}^*(\cdot)))'.$$

Since $h_i(t_i, \mathbf{c}^*(\cdot))$ is continuous for all $i \in N$, $H(t, \mathbf{c}^*(\cdot))$ is also a continuous mapping. Next, let $C([0,1])$ be the sapce of continuous functions ψ mapping from $[0,1]$ to $[0,1]^n$ with the sup norm on $[0,1]$:

$$\|\psi\| = \sup_{v \in [0,1]} |\psi(v)|,$$

where $|\cdot|$ is a norm in R^n and for any $x \in R^n$, $|\cdot|$ is defined as $|x| = \max_{i \in N} |x_i|$. Define

$$M = \{\psi \in C([0,1]) : \|\psi\| \leq 1\}.$$

Notice that $\mathbf{c}^*(\cdot) \in M$. By definition, M is equicontinuous and equibounded. It follows from the Arzela-Ascoli theorem (Burton (2005)) that M is compact. In addition, M is certainly convex by definition.

Define a mapping $T : M \to M$ as follows:

$$(T\mathbf{c}^*)(v) := \int_0^v H(t, \mathbf{c}^*(\cdot))\, dt.$$

We next establish that T is a continuous mapping. To this end, let $\phi, \varphi \in M$. It suffices to show that for any $\epsilon > 0$, there is a δ such that $\|\phi - \varphi\| < \delta$ implies $\|T\phi - T\varphi\| < \epsilon$. Since $h_i\left(t_i, \mathbf{c}^*(\cdot)\right)$ is a continuous mapping from $[0,1] \times [0,1]^n \to [0,1]$, for any $\epsilon > 0$, and $t_i \in [0,1]^n$, there exists an $\delta > 0$ such that $\left|h_i\left(t_i, \phi(\cdot)\right) - h_i\left(t_i, \varphi(\cdot)\right)\right| < \epsilon$ when $\|\phi - \varphi\| < \delta$, for all $i \in N$. To establish the continuity of T, notice that for δ such that $\|\phi - \varphi\| < \delta$, we have:

$$\begin{aligned}
\|T\phi - T\varphi\| &= \sup_{v\in[0,1]} \left| \int_0^v H\left(t, \phi(\cdot)\right) dt - \int_0^v H\left(t, \varphi(\cdot)\right) dt \right| \\
&= \sup_{v\in[0,1]} \left(\max_{i\in N} \left| \int_0^v h_i\left(t_i, \phi(\cdot)\right) dt_i - \int_0^v h_i\left(t_i, \varphi(\cdot)\right) dt_i \right| \right) \\
&\leq \sup_{v\in[0,1]} \left(\max_{i\in N} \int_0^v \left| h_i\left(t_i, \phi(s)\right) - h_i\left(t_i, \varphi(s)\right) \right| dt_i \right) \\
&< \sup_{v\in[0,1]} \left(\max_{i\in N} \int_0^v \epsilon dt_i \right) \\
&= \sup_{v\in[0,1]} (v\epsilon) = \epsilon.
\end{aligned}$$

It follows from the Schauder-Tychonoff fixed-point theorem that T has a fixed point in M. Hence, there exists at least an equlibrium cutoff $\mathbf{c}^*(\cdot)$. This completes the proof.

Proof of Proposition 3

First, notice that $c^*(v)$ is differentiable. Then, we can applying the envelope theorem in equation (5). Notice that $v^k\left(b^k(v)\right) = v$ in a symmetric equilibrium. Differentiating equation (5) derives:

$$\begin{aligned}
c^{*\prime}(v) &= \left[\int_0^1 \int_{c^*(\tau)}^1 k(c,\tau)\, dcd\tau\right]^{n-1} + \sum_{k=1}^{n-1}\left(C_{n-1}^k\left(\int_0^v \int_0^{c^*(\tau)} k(c,\tau)\, dcd\tau\right)^k\left(\int_0^1 \int_{c^*(\tau)}^1 k(c,\tau)\, dcd\tau\right)^{n-1-k}\right), \\
&= \sum_{k=0}^{n-1}\left(C_{n-1}^k\left(\int_0^v \int_0^{c^*(\tau)} k(c,\tau)\, dcd\tau\right)^k\left(\int_0^1 \int_{c^*(\tau)}^1 k(c,\tau)\, dcd\tau\right)^{n-1-k}\right) \\
&= \left(\int_0^v \int_0^{c^*(\tau)} k(c,\tau)\, dcd\tau + \int_0^1 \int_{c^*(\tau)}^1 k(c,\tau)\, dcd\tau\right)^{n-1} \\
&= \left(\int_0^v \int_0^{c^*(\tau)} k(c,\tau)\, dcd\tau + \int_v^1 \int_{c^*(\tau)}^1 k(c,\tau)\, dcd\tau + \int_0^v \int_{c^*(\tau)}^1 k(c,\tau)\, dcd\tau\right)^{n-1} \\
&= \left(\int_0^v \int_0^1 k(c,\tau)\, dcd\tau + \int_v^1 \int_{c^*(\tau)}^1 k(c,\tau)\, dcd\tau\right)^{n-1} \\
&= \left(1 - \int_v^1 \int_0^1 k(c,\tau)\, dcd\tau + \int_v^1 \int_{c^*(\tau)}^1 k(c,\tau)\, dcd\tau\right)^{n-1} \\
&= \left(1 - \int_v^1 \int_0^{c^*(\tau)} k(c,\tau)\, dcd\tau\right)^{n-1}.
\end{aligned}$$

It follows that $c_i^{*\prime}(v)$ is also differentiable and $c^{*\prime}(1) = 1$. Last, integrating equation (8) from 0 to 1 and noticing $c^*(0) = 0$ for all $i \in N$, we derive equation (7). To see why $c_i^*(0) = 0$, notice that $b^k(0) = 0$ and $v^k(0) = 0$, since $v^k(\cdot)$ is the inverse bidding function. It follows that $c^*(0) = 0$. This completes the proof.

Proof of Theorem 2

The existence of the symmetric equilibrium can be established by the Schauder-Tychonoff fixed-point theorem. Here we only need to prove the uniqueness of the symmetric equilibrium. Suppose, by contradiction, that we have two different symmetric equilibria $x(v)$ and $y(v)$. Then we have:

$$x'(v) = \left[1 - \int_v^1 \int_0^{x(\tau)} k(\tau,c)\, dcd\tau\right]^{n-1},$$

$$y'(v) = \left[1 - \int_v^1 \int_0^{y(\tau)} k(\tau,c)\, dcd\tau\right]^{n-1}.$$

Without loss of generality, suppose $x(1) > y(1)$, then by the continuity of $x(v)$ and $y(v)$, there exists a $v^* \in [0,1)$ such that $x(v^*) = y(v^*) := c(v^*)$ and $x(\tau) > y(\tau)$ for all $\tau \in (v^*, 1]$ by noting that $x(0) = y(0)$. Consider two mutually exclusive cases. First, if $k(v,c) > 0$ with positive probability measure on $(v^*, 1) \times (c(v^*), 1)$, then $x(\tau) > y(\tau)$ for all $\tau \in (v^*, 1]$ implies that

$$\int_0^{x(\tau)} k(\tau, c)\, dc > \int_0^{y(\tau)} k(\tau, c)\, dc,$$

for all $\tau \in (v^*, 1)$. It follows that $x'(v^*) < y'(v^*)$, which is a contradiction to $x(\tau) > y(\tau)$ for all $\tau \in (v^*, 1]$. Hence, we must have $x(\tau) = y(\tau)$ for all $\tau \in (v^*, 1]$ in this case. Second, if $k(v,c) > 0$ with zero probability measure on $(v^*, 1) \times (c(v^*), 1)$, then, by the same argument, $x'(\tau) = y'(\tau)$ for all $\tau \in (v^*, 1]$. It follows from $x(v^*) = y(v^*)$ that $x(\tau) = y(\tau)$ for all $\tau \in (v^*, 1]$ in this case. Hence, both cases lead to a contradiction to $x(1) > y(1)$. Therefore, we have established that there exists a $v^* \in [0,1)$, such that $x(\tau) = y(\tau)$ for all $\tau \in [v^*, 1]$. If $v^* = 0$, the proof is complete.

Now, consider an arbitrary closed interval $[\alpha, \beta] \subset [0,1]$. Suppose $x(\alpha) = y(\alpha)$ and $x(\beta) = y(\beta)$; and, without loss of generality, $x(\tau) > y(\tau)$ for all $\tau \in (\alpha, \beta)$. It follows from the argument above that $x'(\tau) < y'(\tau)$ for $\tau \in (\alpha, \beta)$, which is in contradiction to $x(\tau) > y(\tau)$ for all $\tau \in (\alpha, \beta)$. Hence, we have established that $x(v) = y(v)$ for all $v \in [0,1]$. Therefore, the symmetric equilibrium is unique. This completes the proof.

Proof of Proposition 4

First, notice that $F_A = F_B$ implies $f_A = f_B$. Next, we establish that $c_A^*(1) > c_B^*(1)$. Suppose, by contradiction, that $c_A^*(1) \leq c_B^*(1)$. Then, there exists a $v^* \in [0,1)$ such that $c_A^*(v) \leq c_B^*(v)$ for all $v \in (v^*, 1]$. This implies that

$$c_A^{*\prime}(v^*) = \left[1 - \int_{v^*}^1 f(\tau)\, G_A(c_A^*(\tau))\, d\tau\right]^{n-1},$$

$$c_B^{*\prime}(v^*) = \left[1 - \int_{v^*}^1 f(\tau)\, G_B(c_B^*(\tau))\, d\tau\right]^{n-1}.$$

Since $G_A < G_B$ and $c_A^*(v) \leq c_B^*(v)$ for all $v \in (v^*, 1]$, we have $G_A(c_A^*(\cdot)) < G_B(c_B^*(\cdot))$. It follows from that $c_A^{*\prime}(v^*) > c_B^{*\prime}(v^*)$. It follows that there exists a $\bar{v}^* > v^*$ such that $c_A^*(v) > c_B^*(v)$ for all $v \in (v^*, \bar{v}^*)$, leading to a contradiction.

Then, we show $c_A^*(v) > c_B^*(v)$ for all $v \in (0,1]$. Without loss of generality, assume there exists a $v^* \in [0,1)$ such that $c_A^*(v^*) = c_B^*(v^*)$ and $c_A^*(v) > c_B^*(v)$ for all $v \in (v^*, 1]$. This implies that

$c_A^{*\prime}(v^*) > c_B^{*\prime}(v^*)$, which means:

$$\int_{v^*}^{1} f(\tau)\, G_A(c_A^*(\tau))\, d\tau < \int_{v^*}^{1} f(\tau)\, G_B(c_B^*(\tau))\, d\tau.$$

In addition, since $c_A^{*\prime}(v^*) > c_B^{*\prime}(v^*)$, there exists a $\underline{v}^* < v^*$ such that $c_A^*(v) > c_B^*(v)$ for all $v \in (\underline{v}^*, v^*)$ and $c_A^*(\underline{v}^*) = c_B^*(\underline{v}^*)$ by noticing $c_A^*(0) = c_B^*(0)$. This implies that $c_A^{*\prime}(\underline{v}^*) > c_B^{*\prime}(\underline{v}^*)$. To see this, notice that Since $G_A < G_B$ and $c_A^*(v) < c_B^*(v)$ for all $v \in (\underline{v}^*, v^*)$, we have $G_A(c_A^*(\cdot)) < G_B(c_B^*(\cdot))$ for all $v \in (\underline{v}^*, v^*)$, meaning

$$\int_{\underline{v}^*}^{v^*} f(\tau)\, G_A(c_A^*(\tau))\, d\tau < \int_{\underline{v}^*}^{v^*} f(\tau)\, G_B(c_B^*(\tau))\, d\tau.$$

Hence, we have

$$\begin{aligned}
c_A^{*\prime}(\underline{v}^*) &= \left[1 - \int_{\underline{v}^*}^{1} f(\tau)\, G_A(c_A^*(\tau))\, d\tau\right]^{n-1} \\
&= \left[1 - \int_{\underline{v}^*}^{v^*} f(\tau)\, G_A(c_A^*(\tau))\, d\tau - \int_{v^*}^{1} f(\tau)\, G_A(c_A^*(\tau))\, d\tau\right]^{n-1} \\
&> \left[1 - \int_{\underline{v}^*}^{v^*} f(\tau)\, G_B(c_B^*(\tau))\, d\tau - \int_{v^*}^{1} f(\tau)\, G_B(c_B^*(\tau))\, d\tau\right]^{n-1} \\
&= \left[1 - \int_{v^*}^{1} f(\tau)\, G_B(c_B^*(\tau))\, d\tau\right]^{n-1} = c_B^{*\prime}(\underline{v}^*).
\end{aligned}$$

This then implies that there exists a $\tilde{v} \in (\underline{v}^*, v^*)$ such that $c_A^*(v) > c_B^*(v)$ for all $v \in (\underline{v}^*, \tilde{v}) \subset (\underline{v}^*, v^*)$, resulting in a contradiction. Hence, $c_A^*(v) > c_B^*(v)$ for all $v \in (0, 1]$. This completes the proof.

Proof of Proposition 5

We prove this theorem by using the contraction mapping theorem. First, let $C([0,1])$ be the sapce of continuous functions ψ mapping from $[0,1]$ to $[0,1]^2$ with the sup norm on $[0,1]$:

$$\|\psi\| = \sup_{v\in[0,1]} |\psi(v)|,$$

where where $|\cdot|$ is a norm in R^n and for any $x \in R^n$, $|\cdot|$ is defined as $|x| = \max_{i\in N} |x_i|$. Define

$$M = \{\psi \in C([0,1]) : \|\psi\| \leq 1\}.$$

It is certainly that the normed space $(M, \|\cdot\|)$ is a Banach space. Notice that $\mathbf{c}^*(\cdot) = (c_1(\cdot), c_2(\cdot)) \in M$.

Next, define a mapping $h_i(t_i, \mathbf{c}^*(\cdot))$ from $[0,1] \times [0,1]^2 \to [0,1]$, for all $i \in \{1,2\}$ as follows:

$$h_1(t_1, \mathbf{c}^*(\cdot)) = 1 - \int_{v_2(b_1(t_1))}^{1} \int_0^{c_2^*(\tau)} k_2(c,\tau)\, dcd\tau$$

$$h_2(t_2, \mathbf{c}^*(\cdot)) = 1 - \int_{v_1(b_2(t_2))}^{1} \int_0^{c_1^*(\tau)} k_1(c,\tau)\, dcd\tau$$

Define $H(t, \mathbf{c}^*(\cdot)) : [0,1]^2 \times [0,1]^2 \to [0,1]^2$ as follows:

$$H(t, \mathbf{c}^*(\cdot)) = \left(h_1(t_1, \mathbf{c}^*(\cdot)), h_2(t_2, \mathbf{c}^*(\cdot))\right)'.$$

Then, we define a mapping $T : M \to M$ as follows:

$$\begin{aligned}
(T\mathbf{c}^*)(v) &= \int_0^v H(t, \mathbf{c}^*(\cdot))\, dt \\
&= \begin{bmatrix} \int_0^v \left(1 - \int_{v_2(b_1(t_1))}^1 \int_0^{c_2^*(\tau)} k_2(c,\tau)\, dcd\tau\right) dt_1 \\ \int_0^v \left(1 - \int_{v_1(b_2(t_2))}^1 \int_0^{c_1^*(\tau)} k_1(c,\tau)\, dcd\tau\right) dt_2 \end{bmatrix} \\
&= \begin{bmatrix} v - \int_0^v \int_{v_2(b_1(t_1))}^1 \int_0^{c_2^*(\tau)} k_2(c,\tau)\, dcd\tau dt_1 \\ v - \int_0^v \int_{v_1(b_2(t_2))}^1 \int_0^{c_1^*(\tau)} k_1(c,\tau)\, dcd\tau dt_2 \end{bmatrix}
\end{aligned}$$

Since $g_i(\cdot)$ and $f_i(\cdot)$, $i \in \{1,2\}$, are integrable over both arguments and bidding functions are differentiable (hence continuous), $h_i(t_i, \mathbf{c}^*(\cdot))$ and $H(t, \mathbf{c}^*(\cdot))$ are continuous, for all $i \in N$.

It suffices to show that Tc^* is a contraction mapping. To show this, for any $x, y \in M$, where $x = (x_1, x_2)'$ and $y = (y_1, y_2)'$, we have:

$$\begin{aligned}
(Tx)(v) - (Ty)(v) &= \begin{bmatrix} \int_0^v \int_{v_2(b_1(t_1))}^1 \int_0^{y_2(\tau)} k_2(c,\tau)\, dcd\tau dt_1 - \int_0^v \int_{v_2(b_1(t_1))}^1 \int_0^{x_2(\tau)} k_2(c,\tau)\, dcd\tau dt_1 \\ \int_0^v \int_{v_1(b_2(t_2))}^1 \int_0^{y_1(\tau)} k_1(c,\tau)\, dcd\tau dt_2 - \int_0^v \int_{v_1(b_2(t_2))}^1 \int_0^{x_1(\tau)} k_1(c,\tau)\, dcd\tau dt_2 \end{bmatrix} \\
&= \begin{bmatrix} \int_0^v \left(\int_{v_2(b_1(t_1))}^1 \int_{x_2(\tau)}^{y_2(\tau)} k_2(c,\tau)\, dcd\tau\right) dt_1 \\ \int_0^v \left(\int_{v_1(b_2(t_2))}^1 \int_{x_1(\tau)}^{y_1(\tau)} k_1(c,\tau)\, dcd\tau\right) dt_2 \end{bmatrix}.
\end{aligned}$$

Hence, we have

$$
\begin{aligned}
\|(Tx)(v)-(Ty)(v)\| &= \sup_{v\in[0,1]}\max_{i\in\{1,2\}}\left|\int_0^v\int_{v_j(b_i(t_i))}^1\int_{x_j(\tau)}^{y_j(\tau)} k_j(c,\tau)\,dcd\tau dt_i\right| \\
&\leq \sup_{v\in[0,1]}\max_{i\in\{1,2\}}\int_0^v\left(\bar{k}_j\left(1-v_j\left(b_i\left(t_i\right)\right)\right)\sup_{s\in[0,1]}\max_{j\in\{1,2\}}\left|x_j(s)-y_j(s)\right|\right)dt_i \\
&= \|x(s)-y(s)\|\left(\sup_{v\in[0,1]}\max_{i\in\{1,2\}}\left(\int_0^v\bar{k}_j\left(1-v_j\left(b_i\left(t_i\right)\right)\right)dt_i\right)\right) \\
&\leq \|x(s)-y(s)\|\left(\max_{i\in\{1,2\}}\left(\int_0^1\bar{k}_j\left(1-v_j\left(b_i\left(t_i\right)\right)\right)dt_i\right)\right) \\
&< \|x(s)-y(s)\|,
\end{aligned}
$$

where $\bar{k}_j := \sup_{c,v\in[0,1]}k_i(c,v)$. The last inequality holds because $\sup_{c,v\in[0,1]}k_i(c,v) < \frac{1}{\int_0^1[1-v_i(b_j(v))]dv}$ for $i,j\in\{1,2\}$ and $i\neq j$. Therefore, $T: M\to M$ is a contraction mapping. Since M is a complete metric space, it follows from the contraction mapping theorem that T has a unique fixed point in M. Hence, functional equation $\mathbf{c}^* = T\mathbf{c}^*$ has a unique solution.

This completes the proof.

Proof of Corollary 1

Suppose the valuations and participation costs are independently distributed, i.e. $K_i(v_i,c_i) = F_i(v_i)\,G_i(c_i)$ for all $i\in N$. The most of the argument in the proof of Proposition 5 applies here. However, we could have a weaker sufficient condition. To see this, notice that

$$
\begin{aligned}
\|(Tx)(v)-(Ty)(v)\| &= \sup_{v\in[0,1]}\max_{i\in\{1,2\}}\left|\int_0^v\int_{v_j(b_i(t_i))}^1 \left(G_j\left(y_j\left(\tau\right)\right)-G_j\left(x_j\left(\tau\right)\right)\right)f_j\left(\tau\right)d\tau dt_i\right| \\
&\leq \sup_{v\in[0,1]}\max_{i\in\{1,2\}}\int_0^v\int_{v_j(b_i(t_i))}^1 \left|G_j\left(y_j\left(\tau\right)\right)-G_j\left(x_j\left(\tau\right)\right)\right|f_j\left(\tau\right)d\tau dt_i \\
&\leq \sup_{v\in[0,1]}\max_{i\in\{1,2\}}\left(\bar{g}_j\int_0^v\int_{v_j(b_i(t_i))}^1 \left|y_j\left(\tau\right)-x_j\left(\tau\right)\right|f_j\left(\tau\right)d\tau dt_i\right) \\
&\leq \sup_{v\in[0,1]}\max_{i\in\{1,2\}}\left(\bar{g}_j\int_0^v\int_{v_j(b_i(t_i))}^1 \left(\sup_{s\in[0,1]}\max_{j\in\{1,2\}}\int_0^s\left|y_j\left(u\right)-x_j\left(u\right)\right|du\right)f_j\left(\tau\right)d\tau dt_i\right) \\
&= \|x\left(s\right)-y\left(s\right)\|\left(\sup_{v\in[0,1]}\max_{i\in\{1,2\}}\left(\int_0^v\int_{v_j(b_i(t_i))}^1 \bar{g}_jf_j\left(\tau\right)d\tau dt_i\right)\right) \\
&\leq \|x\left(s\right)-y\left(s\right)\|\left(\max_{i\in\{1,2\}}\left(\int_0^1\int_{v_j(b_i(t_i))}^1 \bar{g}_jf_j\left(\tau\right)d\tau dt_i\right)\right) \\
&= \|x\left(s\right)-y\left(s\right)\|\left(\max_{i\in\{1,2\}}\left(\bar{g}_j\int_0^1\left[1-F_j\left(v_j\left(b_i\left(t_i\right)\right)\right)\right]dt_i\right)\right) \\
&< \|x\left(s\right)-y\left(s\right)\|,
\end{aligned}
$$

where $\bar{g}_j := \sup_{c\in[0,1]}g_i\left(c\right)$. The last inequality holds because $\sup_{c\in[0,1]}g_j\left(c\right) < \frac{1}{\int_0^1[1-F(v_j(b_i(v)))]dv}$ for $i,j\in\{1,2\}$ and $i\neq j$. Last, notice that the sufficient conditions when distribution functions are independent and when the sufficient condition in the general case can be rewritten as follows:

$$\int_0^1\int_{v_j(b_i(v))}^1 \bar{g}_jf_j\left(\tau\right)d\tau dv < 1,$$

$$\int_0^1\int_{v_j(b_i(v))}^1 \bar{k}_jd\tau dv < 1.$$

It follows that

$$\int_0^1\int_{v_j(b_i(v))}^1 \bar{g}_jf_j\left(\tau\right)d\tau dv < \int_0^1\int_{v_j(b_i(v))}^1 \bar{k}_jd\tau dv < 1.$$

The sufficient condition when distribution functions are independent holds when the sufficient condition in the general case holds.