Stability and Efficiency in Dynamic Matching with Transfers

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Abstract

This paper introduces a model for multi-period many-to-one matchings with transfers and proposes a solution concept called *individually rational recursive core (IRRC)* in which no coalition from any period has the incentive to deviate from the original plan. We find that when continuous transfers are allowed, the substitutability introduced in this paper is sufficient to guarantee the existence of competitive equilibria and IRRC for underlying economic environments. Furthermore, we show that an IRRC is always efficient.

Keywords: Dynamic matching; Stability; Efficiency; Competitive equilibrium
JEL classification: C78, D61, D71, D78

1 Introduction

In a conventional static two-sided matching problem, it is easy to find a stable matching. Gale and Shapley (1962) were the first to define the solution concept of stability in static one-to-

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one matching and introduced the *deferred acceptance algorithm* that can always lead to a stable outcome in the matching. Hatfield and Milgrom (2005) studied the problem of many-to-one matching with contracts. They proposed a substitutes condition (HM substitutes), and showed that if agents' preferences satisfy HM substitutes, then the set of stable matchings is nonempty. For static labor market matching with transfers, Kelso and Crawford (1982) proposed a salary-adjustment process and demonstrated that, under the KC substitutes condition, the outcome of this process would converge to a core allocation in a discrete market and to a strict core allocation in a continuous market.[1] Hatfield et al. (2013) then extended the model to trading networks and found that, if agents' preferences are fully substitutable by their definition and continuous transfers are allowed, one can always construct a competitive equilibrium (which reduces to the strict core of Kelso and Crawford (1982) when continuous transfers are allowed), and the sets of stable outcomes and competitive equilibria are in a sense equivalent.

These studies only consider static matching, but matching problems in many situations are dynamic rather than static. Also, the multi-period matchings with transfers are common phenomena in practice. For example, in the labor market, the contracts between workers and firms are usually signed by stages. When an original contract expires, the two parties usually need to renew the contract.

In this paper we introduce a dynamic model for many-to-one matchings with transfers, and extend the main results of Hatfield et al. (2013) to the dynamic setting. Such an extension is not trivial, as it requires more involved modifications and proofs. We propose a solution concept called *individually rational recursive core (IRRC)*. We find that, in a dynamic matching with continuous transfers (so that utilities are quasi-linear), if agents' preferences satisfy the substitutability introduced in this paper, one can always construct a competitive equilibrium by the process of Kelso and Crawford (1982) in the dynamic setting. In a competitive equilibrium, given a payment plan for each contract, each agent maximizes his utility. Then the set of IRRC is nonempty and any IRRC outcome is efficient.

Studies on dynamic matching are of course not new. However, all the studies in the existing literature focus on dynamic matching without transfers, and then the outcomes under the solution concepts introduced are in general not efficient. Damiano and Lam (2005) were the first to study dynamic matching. They first defined two solution concepts for stability in a one-to-one dynamic matching setting, called the *core* and *recursive core*, respectively. In the recursive core in a matching, no coalition of individuals has the incentive to deviate from

[1]Discrete market means a market that only allows discrete transfers, and continuous market means a market that allows continuous transfers.

any period, but the recursive core may be empty in general although it is Pareto efficient. To obtain positive results, either weaker solution concepts or stronger assumptions on agents' preferences should be adopted. They then introduced two weaker solution concepts, namely, *self-sustaining stability* and *strict self-sustaining stability* based on the mind of "credibility", and provided conditions for the existence of these two solution concepts.

Kurino (2009, 2020) proposed the notion of *credible group stability* that is also based on the mind of "credibility", and showed that under some conditions imposed, the set of credibly group-stable outcomes is always nonempty. He also defined another solution concept called *one-shot group stability* and showed the existence of dynamic matchings that satisfy one-shot group stability under a sufficient condition. However, these solution concepts are not Pareto efficient.

Other studies on dynamic matching include Bando (2012), Kennes et al. (2014, 2018), Pereyra (2013), Doval (2017), and Kadam and Kotowski (2018).

All in all, none of the existing studies on dynamic matching allows transfers. However, in reality, transfers commonly exist in many dynamic matching problems, typically in labor markets and trading networks, where the payment or trading price is usually taken as a condition for the matching process between firms and workers. Agents would bargain over the payments or prices before signing contracts. Besides, under general dynamic matching environments without transfers, the recursive core proposed by Damiano and Lam (2005) may be empty while there is always an IRRC under the substitutability of preferences and continuity of transfers.

The remainder of this paper is organized as follows. In Section 2, we formalize our model, and propose the substitutes condition. Section 3 introduces the solution concept of IRRC, and shows its existence and efficiency. Section 4 concludes. All proofs are in the appendix.

2 The Model

We begin with a two-sided many-to-one dynamic matching market with continuous transfers. Agents in the economy consist of two disjoint sets W and F, where $W = \{w_1, w_2, \ldots, w_m\}$ is the finite set of workers, and $F = \{f_1, f_2, \ldots, f_n\}$ is the finite set of firms. Generic agents are denoted by $i \in W \cup F$ while generic firms and workers are denoted by f and w, respectively. There are T ($T < \infty$) periods in this matching. We assume that the sets of workers and firms remain unchanged during all periods, that is, no worker or firm will enter or exit the market.

Moreover, we assume that only the agents in the same period can match.

In period t ($t = 1, ..., T$), a typical contract σ^t is a pair $(\theta^t, p(\theta^t))$, where θ^t denotes the job between worker w_{σ^t} and firm f_{σ^t} of the contract, and $p(\theta^t) \in \mathbb{R}$ the purely financial (payment) agreement of the contract. $p(\theta^t)$ can be an arbitrary real number.[2] Let A^t be the set of all possible jobs at t, and A^t is finite. A complete payment vector for all jobs in time t is denoted by $P^t \in \mathbb{R}^{|A^t|}$. The relations between firms and workers at t are governed by the set $\Sigma^t \equiv A^t \times \mathbb{R}^{|A^t|}$ of bilateral contracts. Given $\boldsymbol{A} = (A^1, \ldots, A^T)$, let $\boldsymbol{P} = (P^1, \ldots, P^t) \in \mathbb{R}^{|\boldsymbol{A}|}$ be a complete payment plan for all jobs in all periods.

For a contract $\sigma^t = (\theta^t, p(\theta^t))$ at t, denote by $\alpha(\sigma^t) = \theta^t$ the job involved in σ^t. A trivial contract $\sigma^t_w = \emptyset$ or $\sigma^t_f = \emptyset$ indicates that worker w does not have a job or firm f does not hire any worker at t so that $\alpha(\emptyset) = \emptyset$ and $p(\emptyset) = 0$. We assume that each contract $\sigma^t \in \Sigma^t$ will be executed in the current period, and in the next period, the agents involved in Σ^t should sign new contracts. That is, no contract can be inter-temporal. In each period, a worker can only choose one contract, and a firm can choose a set of contracts. We further assume that there is no commitment in any contract in each period, that is, any agent can deviate from the current plan as he wants.

For a set of contracts $X^t \subseteq \Sigma^t$ at t, denote by $a(X^t)$ the set of agents involved in X^t, and X^t_i the set of contracts in X^t that involve agent i. Let $\boldsymbol{\Sigma} = (\Sigma^1, \ldots, \Sigma^T)$, $\boldsymbol{X} = (X^1, \ldots, X^T) \subseteq \Sigma$ [3], and $\boldsymbol{X}_i = (X^1_i, \ldots, X^T_i)$. Analogously, for any $\Psi^t \subseteq A^t$, let $a(\Psi^t)$ be the set of agents involved in Ψ^t, Ψ^t_i be the subset in Ψ^t that involves agent i, and $\boldsymbol{\Psi}_i = (\Psi^1_i, \ldots, \Psi^T_i)$. Note that $a(\boldsymbol{X}) = \bigcup_{t=1}^T a(X^t)$ and $a(\boldsymbol{\Psi}) = \bigcup_{t=1}^T a(\Psi^t)$, where $\boldsymbol{\Psi} = (\Psi^1, \ldots, \Psi^T)$.

An *arrangement* at time t is a pair $[\Psi^t, P^t]$, where $\Psi^t \subseteq A^t$ and $P^t \in \mathbb{R}^{|A^t|}$. For any arrangement $[\Psi^t, P^t]$, we denote by $\kappa([\Psi^t, P^t]) \equiv \bigcup_{\theta^t \in \Psi^t} (\theta^t, p(\theta^t))$ [4] the set of contracts induced by the arrangement in time t. Let $\boldsymbol{\kappa}([\boldsymbol{\Psi}, \boldsymbol{P}]) = (\kappa([\Psi^1, P^1]), ..., \kappa([\Psi^T, P^T]))$. Note that $\alpha(\kappa([\Psi^t, P^t])) = \Psi^t$ and $\alpha(\boldsymbol{\kappa}([\boldsymbol{\Psi}, \boldsymbol{P}])) = \boldsymbol{\Psi}$.

Definition 1 We say that a set of contracts $X^t \subseteq \Sigma^t$ at t ($t = 1, ..., T$) is *feasible* if:

(a) for each $w \in W$, either $X^t_w = \emptyset$, or $|X^t_w| = 1$;

(b) there is no job θ^t and payments $p(\theta^t) \neq \widehat{p}(\theta^t)$ such that both contracts $(\theta^t, p(\theta^t))$ and $(\theta^t, \widehat{p}(\theta^t))$ are in X^t.

That is, if a set of contracts at t ($t = 1, ..., T$) is feasible, then a worker can choose at most one contract while a firm can hire more than one worker; and each job can only be associated

[2] This means transfers are continuous.

[3] Here we define $\boldsymbol{X} = (X^1, \ldots, X^T) \subseteq \Sigma$ if for any $t = 1, ..., T$, $X^t \subseteq \Sigma^t$.

[4] Here $p(\theta^t)$ is the payment of θ^t in the payment vector P^t.

with at most one payment in the set.

We say that a set of jobs $\Psi^t \subseteq A^t$ at t $(t = 1,...,T)$ is *feasible* if for any $P^t \in \mathbb{R}^{|A^t|}$, $\kappa[\Psi^t, P^t]$ is feasible.

An *outcome* is a sequence $\mathbf{\Omega} = (\Omega^1, \ldots, \Omega^T)$, where Ω^t is a feasible set of contracts in period $t = 1,...,T$. Denote by $\mathbf{\Omega}_i = (\Omega_i^1, \ldots, \Omega_i^T)$ agent i's outcome, and $\mathbf{\Omega}_S = (\Omega_S^1, \ldots, \Omega_S^T)$ group (called *coalition*) S's outcome in the outcome $\mathbf{\Omega}$.

A *history* at t is a path of feasible contracts before t, denoted by $h^t := (\Omega^0, \Omega^1 \ldots, \Omega^{t-1})$ with $h^1 := \Omega^0 \equiv \emptyset$. Let $\mathscr{H}^t$ be the set of all histories at t, and $\mathscr{H} := \bigcup_{t=0}^{T} \mathscr{H}^t$ be the set of all histories. Given $h^t = (\Omega^0, \Omega^1 \ldots, \Omega^{t-1})$, denote by $\mathbf{\Omega}^{\geq t} = (\Omega^t, ..., \Omega^T)$ the continuation outcome from period t $(t = 1,...,T)$. Note that $\mathbf{\Omega} = \mathbf{\Omega}^{\geq 1}$. Similarly, let $\boldsymbol{A}^{\geq t} = (A^t, ..., A^T)$, $\boldsymbol{P}^{\geq t} = (P^t, ..., P^T)$, $\mathbf{\Psi}^{\geq t} = (\Psi^t, ..., \Psi^T)$ and $\boldsymbol{X}^{\geq t} = (X^t, ..., X^T)$.

2.1 Preferences

Each agent i' preference at period t $(t = 1,...,T)$ is defined by the cardinal utility function u_i^t over the collection of sets of contracts $\{\Omega_i^t\}$ in Σ^t and is quasi-linear with respect to the associated transfers. Specifically, for each t $(t = 1,...,T)$, agent $i \in W$'s utility from a contract $\Omega_i^t = \{\sigma_i^t\} = (\theta^t, p(\theta^t))$ is:

$$u_i^t(\Omega_i^t) = u_i^t(\sigma_i^t) = v_i^t(\theta^t) + p(\theta^t); \tag{1}$$

and agent $i \in F$'s utility from a set of contracts Ω_i^t is:

$$u_i^t(\Omega_i^t) = v_i^t(\alpha(\Omega_i^t)) - \sum_{(\theta^t, p(\theta^t)) \in \Omega_i^t} p(\theta^t), \tag{2}$$

where $v_i^t(\cdot)$ is assumed to be bounded for all $i \in W \cup F$ and $t = 1,...,T$.[5]

Agent i's discounted utility of an outcome $\mathbf{\Omega} = (\Omega^1, \ldots, \Omega^T)$ then is:

$$U_i(\mathbf{\Omega}_i) = \sum_{t=1}^{T} \delta^{t-1} u_i^t(\Omega_i^t), \tag{3}$$

where δ is the discount factor that is the same for all agents.

[5]Even if $v_i^t(\cdot)$ is unbounded, in Section 3 when we show the existence of competitive equilibrium, as Hatfield et al. (2013) pointed out, we can transform each agent's utility function to a bounded one, construct a competitive equilibrium, and then transform the competitive equilibrium back into the original economy. Therefore, all the results in this paper will not change.

Given $h^t = (\Omega^0, \Omega^1 \ldots, \Omega^{t-1})$, we can also define agent i's continuation discounted utility of $\mathbf{\Omega}^{\geq t} = (\Omega^t, \ldots, \Omega^T)$ from period t $(t = 1, ..., T)$ as

$$U_i^t(\mathbf{\Omega}_i^{\geq t}) = \sum_{s=t}^{T} \delta^{s-t} u_i^s(\Omega_i^s). \tag{4}$$

It is clear that $U_i(\mathbf{\Omega}_i) = U_i^1(\mathbf{\Omega}_i^{\geq 1})$.

Given $\boldsymbol{X}^{\geq t} \subseteq \mathbf{\Sigma}^{\geq t}$, the continuation choice correspondence of agent i from period t $(t = 1, ..., T)$ is defined as the collection of sets of contracts maximizing the utility of agent i:

$$C_i(\boldsymbol{X}^{\geq t}) \equiv \arg \max_{\boldsymbol{Y}^{\geq t} \subseteq \boldsymbol{X}_i^{\geq t}, \boldsymbol{Y}^{\geq t} \text{ is feasible}} U_i^t(\boldsymbol{Y}^{\geq t}).[6] \tag{5}$$

Given a payment plan $\boldsymbol{P}^{\geq t} \in \mathbb{R}^{|\boldsymbol{A}^{\geq t}|}$, the continuation demand correspondence of agent i from period t $(t = 1, ..., T)$ is defined as the collection of sets of jobs maximizing the utility of agent i under prices $\boldsymbol{P}^{\geq t}$:

$$D_i(\boldsymbol{P}^{\geq t}) \equiv \arg \max_{\mathbf{\Psi}^{\geq t} \subseteq \boldsymbol{A}_i^{\geq t}, \mathbf{\Psi}^{\geq t} \text{ is feasible}} U_i^t(\kappa[\mathbf{\Psi}^{\geq t}, \boldsymbol{P}^{\geq t}]). \tag{6}$$

For simplicity of notation, we may express $C_i(\boldsymbol{X}^{\geq 1})$ and $D_i(\boldsymbol{P}^{\geq 1})$ as $C_i(\boldsymbol{X})$ and $D_i(\boldsymbol{P})$, respectively.

Definition 2 (Competitive Equilibrium (CE)) We say an arrangement $[\mathbf{\Psi}, \boldsymbol{P}]$ is a *competitive equilibrium* if for all $i \in W \cup F$, $\mathbf{\Psi}_i \in D_i(\boldsymbol{P})$.

It is clear that at an outcome induced by a competitive equilibrium $[\mathbf{\Psi}, \boldsymbol{P}]$, every agent is optimizing given a payment plan $\boldsymbol{P}$.

Definition 3 (Efficient Outcome) We say an outcome $\mathbf{\Omega}$ is *efficient* if for any outcome $\mathbf{\Omega}' \neq \mathbf{\Omega}$, $\sum_{i \in W \cup F} U_i(\mathbf{\Omega}_i) \geq \sum_{i \in W \cup F} U_i(\mathbf{\Omega}'_i)$.

Remark 1 When $U_i(\mathbf{\Omega}_i)$ is maximized, then agent i is optimizing at the outcome $\mathbf{\Omega}$.

2.2 The Substitutes Conditions

In static many-to-one matchings, to find a stable matching, some types of substitutes conditions have been imposed on the preferences of agents, or stable matching may not exist.

[6] We say $\boldsymbol{Y}^{\geq t} = (Y^t, ..., Y^T)$ is feasible if each Y^s $(s = t, ..., T)$ is feasible. Similarly, $\mathbf{\Psi}^{\geq t} = (\Psi^t, ..., \Psi^T)$ is feasible if each Ψ^s $(s = t, ..., T)$ is feasible.

Hatfield and Milgrom (2005) proposed a substitutes condition called HM substitutes and showed that if the preferences of agents satisfy HM substitutes, then there is a stable matching in a many-to-one matching economy. In Hatfield and Kojima (2010), they introduced two weaker substitutes conditions (we call them HK bilateral substitutes and HK unilateral substitutes, respectively), and showed that if these two substitutes conditions are satisfied, there exists a stable matching in a many-to-one matching economy. To study the stability and efficiency in dynamic matchings, we propose a similar substitutes condition, called **substitutability**, for many-to-one dynamic matchings below. This substitutes condition is an extension of HM substitutes to the dynamic case.

For any $\boldsymbol{X} \subseteq \boldsymbol{\Sigma}$, define $Ch_i(\boldsymbol{X}) = \bigcup_{\boldsymbol{Y}_i \in C_i(\boldsymbol{X})} \boldsymbol{Y}_i$, $Ch_W(\boldsymbol{X}) = \bigcup_{w \in W} Ch_w(\boldsymbol{X})$, and $Ch_F(\boldsymbol{X}) = \bigcup_{f \in F} Ch_f(\boldsymbol{X})$. Let $R_i(\boldsymbol{X}) = \boldsymbol{X} - Ch_i(\boldsymbol{X}) =$, $R_W(\boldsymbol{X}) = \boldsymbol{X} - Ch_W(\boldsymbol{X})$ and $R_F(\boldsymbol{X}) = \boldsymbol{X} - Ch_F(\boldsymbol{X})$.

Definition 4 (Choice-Language Substitutability) We say the preference of agent i is *choice-language substitutable* if for any $\boldsymbol{X}' \subseteq \boldsymbol{X}''$ and $\sigma^s \in \boldsymbol{X}'^s \subseteq \boldsymbol{X}''^s$ $(s = 1, ..., T)$, $\sigma^s \in Ch_i^s(\boldsymbol{X}'')$ implies $\sigma^s \in Ch_i^s(\boldsymbol{X}')$.[7]

Choice-language substitutability shows that, given $\boldsymbol{X}' \subseteq \boldsymbol{\Sigma}$, if a contract σ^s at time s $(s = 1, ..., T)$ is not involved in the choice correspondence of agent i, then even if we expand the choice set of agent i's contracts for each period, σ^s will still be rejected by agent i in any of his choice plan. When $T = 1$, it reduces to HM substitutes and is equivalent to KC substitutes (in Kelso and Crawford (1982)).

Definition 5 (Demand-Language Substitutability) For a dynamic matching economy, we say:

(a) the preference of firm f is *demand-language substitutable* if: $\theta_w^s \in \Theta_f^s$ $(s = 1, ..., T)$, and $\boldsymbol{\Theta}_f = (\Theta_f^1, \ldots, \Theta_f^T) \in D_f(\boldsymbol{P})$, whenever the payments of some other jobs increase, making the payment plan change from $\boldsymbol{P}$ to $\widetilde{\boldsymbol{P}}$ $(\widetilde{\boldsymbol{P}} \geq \boldsymbol{P})$, there exists a $\widetilde{\Theta}_f^s$ such that $\theta_w^s \in \widetilde{\Theta}_f^s$, and

$$\widetilde{\boldsymbol{\Theta}}_f = (\widetilde{\Theta}_f^1, \ldots, \widetilde{\Theta}_f^T) \in D_f(\widetilde{\boldsymbol{P}}).$$

(b) the preference of worker w is *demand-language substitutable* if: $\theta_f^s \in \Theta_w^s$ $(s = 1, ..., T)$, and $\boldsymbol{\Theta}_w = (\Theta_w^1, \ldots, \Theta_w^T) \in D_w(\boldsymbol{P})$, whenever the payments of

[7] If $\sigma_i^s = \emptyset$, then $\sigma_i^s \in Ch_i^s(\boldsymbol{X}'')$ still implies $\sigma_i^s \in Ch_i^s(\boldsymbol{X}')$.

some other jobs decrease, making the payment plan change from $\boldsymbol{P}$ to $\widetilde{\boldsymbol{P}}$ ($\widetilde{\boldsymbol{P}} \leq \boldsymbol{P}$), there exists a $\widetilde{\Theta}_w^s$ such that $\theta_f^s \in \widetilde{\Theta}_w^s$, and

$$\widetilde{\boldsymbol{\Theta}}_w = (\widetilde{\Theta}_w^1, \ldots, \widetilde{\Theta}_w^T) \in D_w(\widetilde{\boldsymbol{P}}).$$

Demand-language substitutability shows that, given $\boldsymbol{P}$,

(a) for any firm f, if a job θ^s at time s $(s = 1, ..., T)$ is involved in f's demand correspondence, then whenever the payments of some other jobs increase, θ^s will still be involved in f's demand correspondence;

(b) for any worker w, if a job θ^s at time s $(s = 1, ..., T)$ is involved in w's demand correspondence, then whenever the payments of some other jobs decrease, θ^s will still be involved in w's demand correspondence.

We then have the following proposition whose proof will be given in the appendix.

Proposition 1 *For economies under consideration, choice-language substitutability is equivalent to demand-language substitutability.*

From this proposition, in the following, we simply call the choice-language/demand-language substitutability as substitutability.

By this substitutes condition, we can discuss the existence of competitive equilibrium and IRRC for the underlying dynamic matching problems in the following sections.

3 IRRC and Its Properties

In this section, we introduce the solution concept of *individually rational recursive core (IRRC)* and show some of its properties which extend the main results of Hatfield et al. (2013) to the dynamic setting. We first introduce the solution concept of IRRC. Then we show that competitive equilibrium exists when preferences are substitutable and continuous transfers are allowed. Next we characterize the structure of the set of competitive equilibria under substitutability. Finally, we show the existence and efficiency of IRRC.

3.1 Individually Rational Recursive Core

In this subsection, we propose the following solution concept for stability in dynamic matching.

Definition 6 (IRRC) We say an outcome $\mathbf{\Omega} \subseteq \mathbf{\Sigma}$ is in the *individually rational recursive core* if for any $\mathbf{\Omega}^{\geq t}$ and $t = 1, ..., T$, it satisfies the following condition:
for any coalition $S \subseteq W \cup F$, there is no plan $\widehat{\mathbf{\Omega}}_S^{\geq t} \neq \mathbf{\Omega}_S^{\geq t}$ such that $U_i^t(\widehat{\mathbf{\Omega}}_{iS}^{\geq t}) \geq U_i^t(\mathbf{\Omega}_{iS}^{\geq t})$ for all $i \in S$, and $U_j^t(\widehat{\mathbf{\Omega}}_{jS}^{\geq t}) > U_j^t(\mathbf{\Omega}_{jS}^{\geq t})$ for at least $j \in S$, where $\mathbf{\Omega}_{iS}^{\geq t} = \mathbf{\Omega}_i^{\geq t} \cap \mathbf{\Omega}_S^{\geq t}$.

IRRC is a stronger version of the recursive core in Damiano and Lam (2005). When the matching is one-to-one and agents' preferences are strict, it will reduce to the recursive core defined in Damiano and Lam (2005). In IRRC, no coalition from any period has the incentive to deviate from the original plan. Moreover, any outcome in IRRC is Pareto efficient by definition.

Definition 7 (Pareto Efficient Outcome (PE)) We say an outcome $\mathbf{\Omega}$ is *Pareto efficient* if there is no $\mathbf{\Omega}' \neq \mathbf{\Omega}$ such that $U_i(\mathbf{\Omega}_i') \geq U_i(\mathbf{\Omega}_i)$ for all $i \in W \cup F$, and $U_j(\mathbf{\Omega}_j') > U_j(\mathbf{\Omega}_j)$ for at least one $j \in W \cup F$.

Remark 2 By the form of agents' utilities, we can see efficiency implies Pareto efficiency here, but the converse may not be true.

3.2 Existence of Competitive Equilibrium

To show the existence of IRRC, we first show the existence of competitive equilibrium. The way adopted here to find competitive equilibrium is analogous to a similar finding of Kelso and Crawford (1982), but it requires a more involved proof. For each job θ^s in period s, let $a(\theta^s) = (w, f)$. Suppose that agents' preferences satisfy substitutability. We will prove that there always exists a payment $p^0(\theta^s)$ such that

$$U_w(\Omega_w^1, \ldots, \{(\theta^s, p^0(\theta^s))\}, \ldots, \Omega_w^T) = U_w(\Omega_w^1, \ldots, \emptyset^s, \ldots, \Omega_w^T).$$

This means $p^0(\theta^s)$ is the minimal payment of job θ^s that worker w will accept in period s. If the payment is less than $p^0(\theta^s)$, worker w will prefer to have no job. For any θ^s, let $p(\theta^s)(0) = p^0(\theta^s)$, and we say $p(\theta^s)(0)$ is the permitted payment of θ^s in round 0. Then we can find a competitive equilibrium under substitutability by the following process.

R1. Given $\boldsymbol{A}$, firms begin with a permitted payment plan $\boldsymbol{P}(0)$, where each element in $\boldsymbol{P}(0)$ is $p(\theta^s)(0) = p^0(\theta^s)$, if $\theta^s \in A^s$. Permitted payment $p(\theta^s)(k)$ will be constant at round k except as noted below. In round 0, each firm makes offers to the workers, and we suppose this is costless.

R2. At round k, each firm will choose a most favorable plan within $\kappa(\boldsymbol{A}, \boldsymbol{P}(k))$, and make offers to the involved workers in this plan. Each firm has a unique tie-breaking rule between plans. If agents' preferences satisfy substitutability, then any offer that has not been rejected in round $k-1$ must be repeated in round k, and the payment of this offer must remain constant. If a firm $f \in F$ does not want to hire any worker in some period in its most favorable plan, then in this period, it will choose $\emptyset$.

R3. Each worker will choose a most favorable plan among all the offers he received at round k, and reject the others. Each worker also has a unique tie-breaking rule between plans.

R4. The offer that has not been rejected in the previous round must remain constant. If in round $k-1$, worker w rejects an offer $(\theta^s, p(\theta^s)(k-1))$ from firm f, set $p(\theta^s)(k) = p(\theta^s)(k-1) + \varepsilon$ (where $\varepsilon > 0$), otherwise $p(\theta^s)(k) = p(\theta^s)(k-1)$. Then firms continue to make offers to workers according to their most favorable plans.

R5. The process stops in some round when there is no rejection. Workers will then accept all the offers that they have not rejected.

Through the above process, we then have the following theorem that shows that a competitive equilibrium always exists provided agents' preferences satisfy substitutability. All the proofs of theorems will be given in the appendix.

Theorem 1 *Suppose agents' preferences satisfy substitutability. Then, one can always find a corresponding payment plan $\boldsymbol{P}$ such that there exists a job plan $\boldsymbol{\Theta} \subseteq \boldsymbol{A}$, making the arrangement $[\boldsymbol{\Theta}, \boldsymbol{P}]$ a competitive equilibrium.*

In the following, we will first show some properties of competitive equilibrium.

3.3 Properties of Competitive Equilibrium

We can obtain analogues of the first and second welfare theorems in our dynamic setting.

Theorem 2 *If $[\boldsymbol{\Theta}, \boldsymbol{P}]$ is a competitive equilibrium, then $\boldsymbol{\Theta}$ is efficient, i.e., for any $\boldsymbol{\Theta}' \subseteq \boldsymbol{A}$, we have*

$$\sum_{s=1}^{T} \sum_{i \in W \cup F} \delta^{s-1} v_i^s(\Theta_i^s) \geq \sum_{s=1}^{T} \sum_{i \in W \cup F} \delta^{s-1} v_i^s(\Theta_i'^s). \tag{7}$$

We say a feasible job plan $\boldsymbol{\Psi}$ is efficient if there is a $\boldsymbol{P}$, such that $\kappa[\boldsymbol{\Psi}, \boldsymbol{P}]$ is efficient. The next result provides a converse to Theorem 2, which can be viewed as a version of the second welfare theorem for our dynamic setting: For any efficient job plan $\boldsymbol{\Psi}$ and any competitive equilibrium price vector $\boldsymbol{P}$, the arrangement $[\boldsymbol{\Psi}, \boldsymbol{P}]$ is a competitive equilibrium.

Theorem 3 *Suppose agents' preferences are substitutable. Then for any competitive equilibrium* $[\boldsymbol{\Theta}, \boldsymbol{P}]$ *and efficient job plan* $\boldsymbol{\Psi} \subseteq \boldsymbol{A}$*,* $[\boldsymbol{\Psi}, \boldsymbol{P}]$ *is also a competitive equilibrium.*

Theorem 4 below shows that the set of such vectors is a lattice.

Theorem 4 *Suppose agents' preferences satisfy substitutability. If* $[\boldsymbol{\Theta}, \boldsymbol{P}]$ *is a competitive equilibrium, then the set of competitive equilibrium payment plans is a lattice.*

3.4 The Existence and Efficiency of IRRC

In this subsection, we show the existence and efficiency of IRRC by the competitive equilibrium. The following theorem shows the existence of IRRC.

Theorem 5 *Suppose agents' preferences satisfy substitutability. Then IRRC is nonempty.*

The following theorem shows the efficiency of IRRC.

Theorem 6 *Any IRRC outcome is efficient.*

Theorem 5 and Theorem 6 can be showed by the existence and efficiency of competitive equilibria. When agents' preferences satisfy substitutability, then a competitive equilibrium always exists. We can use the competitive equilibrium to construct an IRRC and get its efficiency.

4 Conclusion

In this paper we have introduced a solution concept called IRRC, for a multi-period many-to-one dynamic matching with continuous transfers. If agents' preferences are substitutable, one can always find a competitive equilibrium and get an IRRC by the competitive equilibrium. Moreover, any IRRC outcome is efficient.

Appendix: Proofs

Proof of Proposition 1: Let $\overline{p}^s$ be a sufficiently large payment at period s such that no firm hires any worker under this payment. Given a payment plan $\boldsymbol{P}$, a firm f will choose a set of jobs as follows:

$$D_f(\boldsymbol{P}) = \arg \max_{\boldsymbol{\Theta} \subseteq \boldsymbol{A}_f, \boldsymbol{\Theta} \text{ is feasible}} U_f(\kappa[\boldsymbol{\Theta}, \boldsymbol{P}]). \tag{8}$$

Since the firm is utility-maximizing, the lower a worker's payment is, the better; the firm does not distinguish an unavailable worker from an available worker who asks for too high payment. For any $\boldsymbol{X} \subseteq \boldsymbol{\Sigma}$, a payment vector at s is then defined as:

$$\widehat{P}^s(X^s) = \min\left\{(p(\theta_1^s), ..., p(\theta_{|A^s|}^s)) : p(\theta_r^s) = \overline{p}^s \text{ or } (\theta_r^s, p(\theta_r^s)) \in X^s\right\},$$

where $r = 1, ..., |A^s|$. Let $\widehat{\boldsymbol{P}}(\boldsymbol{X}) = (\widehat{P}^1(X^1), \ldots, \widehat{P}^T(X^T))$. From the perspective of a profit-maximizing firm, having a set of contracts $\boldsymbol{X}$ available is equivalent to facing a payment plan $\widehat{\boldsymbol{P}}(\boldsymbol{X})$. We first prove the following lemma.

Lemma 1 *For any $\boldsymbol{X}' \subseteq \boldsymbol{X}''$, we have $\widehat{\boldsymbol{P}}(\boldsymbol{X}') \geq \widehat{\boldsymbol{P}}(\boldsymbol{X}'')$. Moreover, given a contract $\sigma_w^s = (\theta_w^s, p(\theta_w^s)) \in X'^s$, if $\theta_w^s \in \Theta_f^s$, $\boldsymbol{\Theta}_f = (\Theta_f^1, \ldots, \Theta_f^T) \in D_f(\widehat{\boldsymbol{P}}(\boldsymbol{X}''))$, and $\sigma_w^s \in \kappa[\boldsymbol{\Theta}_f, \widehat{\boldsymbol{P}}(\boldsymbol{X}'')]$, we have $p(\theta_w^s) = \widehat{P}^s(X''^s(\theta_w^s)) = \widehat{P}^s(X'^s(\theta_w^s))$, where $\widehat{P}^s(X^s(\theta_w^s))$ is the payment of job θ_w^s in the payment vector $\widehat{P}^s(X^s)$ for all $\boldsymbol{X} \subseteq \boldsymbol{\Sigma}$ and θ_w^s.*

The proof of the first part of the lemma is obvious. For the second part, since $\theta_w^s \in \Theta_f^s$, $\boldsymbol{\Theta}_f = (\Theta_f^1, \ldots, \Theta_f^T) \in D_f(\widehat{\boldsymbol{P}}(\boldsymbol{X}''))$ and $\sigma_w^s \in \kappa[\boldsymbol{\Theta}_f, \widehat{\boldsymbol{P}}(\boldsymbol{X}'')]$, we have $p(\theta_w^s) = \widehat{P}^s(X''^s(\theta_w^s))$; otherwise, the firm f can choose a lower payment for job θ_w^s, a contradiction. If $\widehat{P}^s(X'^s(\theta_w^s)) > \widehat{P}^s(X''^s(\theta_w^s))$, given $\sigma_w^s = (\theta_w^s, p(\theta_w^s)) \in X'^s$, then for any $\theta_w^s \in \Theta_f^s$ and $\boldsymbol{\Theta}_f = (\Theta_f^1, \ldots, \Theta_f^T) \in D_f(\widehat{\boldsymbol{P}}(\boldsymbol{X}''))$, we have $\sigma_w^s \notin \kappa[\boldsymbol{\Theta}_f, \widehat{\boldsymbol{P}}(\boldsymbol{X}'')]$, a contradiction. The lemma is proved.

By Lemma 1, we know:

$$\theta_w^s \in \Theta_f^s, \boldsymbol{\Theta}_f = (\Theta_f^1, \ldots, \Theta_f^T) \in D_f(\widehat{\boldsymbol{P}}(\boldsymbol{X})),$$
$$\text{and } \sigma_w^s = (\theta_w^s, p(\theta_w^s)) \in \kappa[\boldsymbol{\Theta}_f, \widehat{\boldsymbol{P}}(\boldsymbol{X})]$$

$$\Leftrightarrow \sigma_w^s = (\theta_w^s, p(\theta_w^s)) \in Ch_f^s(\boldsymbol{X}).$$

Now we show that for any firm $f \in F$, choice-language substitutability is equivalent to demand-language substitutability.

We first show that demand-language substitutability implies choice-language substitutability. Suppose by way of contradiction that demand-language substitutability holds, but choice-language substitutability does not hold for some firm $f \in F$. Then, for some $\boldsymbol{X}' \subseteq \boldsymbol{X}''$, there exists $\sigma_w^s = (\theta_w^s, p(\theta_w^s)) \in X'^s \subseteq X''^s$, such that $\sigma_w^s \in R_f^s(\boldsymbol{X}')$ but $\sigma_w^s \in Ch_f^s(\boldsymbol{X}'')$. This implies $\theta_w^s \in \Theta_f^s$, $\boldsymbol{\Theta}_f = (\Theta_f^1, \ldots, \Theta_f^T) \in D_f(\widehat{\boldsymbol{P}}(\boldsymbol{X}''))$ and $\sigma_w^s \in \kappa[\boldsymbol{\Theta}_f, \widehat{P}(\boldsymbol{X}'')]$. By Lemma 1, $p(\theta_w^s) = \widehat{P}^s(X''^s(\theta_w^s)) = \widehat{P}^s(X'^s(\theta_w^s))$. Then, demand-language substitutability implies that $\sigma_w^s \in Ch_f^s(\boldsymbol{X}')$ since $\widehat{\boldsymbol{P}}(\boldsymbol{X}') \geq \widehat{\boldsymbol{P}}(\boldsymbol{X}'')$, a contradiction.

Conversely, suppose that choice-language substitutability holds for firm $f \in F$. For any two payment plans, $\widetilde{\boldsymbol{P}} \geq \boldsymbol{P}$, in which $\widetilde{p}(\theta_w^s) = p(\theta_w^s)$ for some θ_w^s. Let $\boldsymbol{X}' = \kappa[\boldsymbol{A}, \widetilde{\boldsymbol{P}}]$ and $\boldsymbol{X}'' = \kappa[\boldsymbol{A}, \widetilde{\boldsymbol{P}}] \cup \kappa[\boldsymbol{A}, \boldsymbol{P}]$. Then, $\boldsymbol{X}' \subseteq \boldsymbol{X}''$, $D_f(\boldsymbol{P}) = D_f(\widehat{\boldsymbol{P}}(\boldsymbol{X}''))$ and $D_f(\widetilde{\boldsymbol{P}}) = D_f(\widehat{\boldsymbol{P}}(\boldsymbol{X}'))$. Let $\sigma_w^s = (\theta_w^s, p(\theta_w^s)) \in \kappa[\boldsymbol{\Theta}_f, \widehat{\boldsymbol{P}}(\boldsymbol{X}'')]$. Then $\sigma_w^s \in X'^s \subseteq X''^s$ since $\widetilde{p}(\theta_w^s) = p(\theta_w^s)$. If $\theta_w^s \in \Theta_f^s$, $\boldsymbol{\Theta}_f = (\Theta_f^1, \ldots, \Theta_f^T) \in D_f(\boldsymbol{P}) = D_f(\widehat{\boldsymbol{P}}(\boldsymbol{X}''))$, then $\sigma_w^s \in Ch_f^s(\boldsymbol{X}'')$. By choice-language substitutability, $\sigma_w^s \in Ch_f^s(\boldsymbol{X}')$. Then there exists a $\widehat{\Theta}_f^s$ such that $\theta_w^s \in \widehat{\Theta}_f^s$, and $\widehat{\boldsymbol{\Theta}}_f = (\widehat{\Theta}_f^1, \ldots, \widehat{\Theta}_f^T) \in D_f(\widetilde{\boldsymbol{P}}) = D_f(\widehat{\boldsymbol{P}}(\boldsymbol{X}'))$. Therefore, demand-language substitutability holds.

By symmetry, we can show analogously that for any $w \in W$ that is utility-maximizing, choice-language substitutability is equivalent to demand-language substitutability. Thus, the proposition is proved.

Proof of Theorem 1: We prove this theorem by the following lemmas.

Lemma 2 *When transfers are discrete, after a finite number of rounds, the process will stop if agents' preferences satisfy substitutability.*

Proof. Since agents' preferences satisfy substitutability, if a firm does not want to hire any worker in some periods in its most favorable plan in round 0, then it will still not want to hire any worker in these periods throughout the process. If no firm wants to hire any worker in any period in round 0, then the process stops immediately. Since the set $\boldsymbol{A}$ is finite in each round and the utility $v_i^t(\cdot)$ $(t = 1, ..., T)$ of each agent $i \in W \cup F$ is bounded, then for any job θ^t, there must exist a $\bar{p}(\theta^t)$ such that the firm prefers not offering the contract $(\theta^t, \bar{p}(\theta^t))$ to the worker. Let $\bar{p} = \max_{t \in \{1,,,.T\}, \theta^t \in A^t} \{\bar{p}(\theta^t)\}$. We can see that $\bar{p}$ is bounded, and then the number of rounds of the process cannot exceed $N = [\frac{\bar{p}}{\varepsilon}] + 1$.

Lemma 3 *The process will converge to an individually rational outcome if agents' preferences satisfy substitutability.*

Proof. Since agents' preferences satisfy substitutability, if a firm does not want to hire any worker in some periods in its most favorable plan in round k $(k = 0, 1, ...)$, then it will still not want to hire any worker in these periods in the following rounds. Then any firm is individually rational in the final outcome. At round 0, the permitted payment plan $\boldsymbol{P}(0)$ contains all the minimal payments that the workers will accept. By substitutability, it can be seen that throughout the process, the workers can only be improved from round 0 to the end, and cannot be worse off in any round. Then any worker is individually rational in the final outcome.

Hence the final outcome must be individually rational.

Now we show that when transfers are continuous, if agents' preferences satisfy substitutability, there is a competitive equilibrium.

Proof. Suppose by way of contradiction that there is no competitive equilibrium. We shall argue that for sufficiently small choices of the unit of measurement ε, this implies that any individually rational arrangement cannot be a final outcome of the process in the corresponding discrete market.

Consider any arrangement $[\boldsymbol{\Theta}, \boldsymbol{P}]$ such that $\kappa[\boldsymbol{\Theta}, \boldsymbol{P}]$ is individually rational. Let $\kappa[\boldsymbol{\Theta}, \boldsymbol{P}] = [\boldsymbol{\Theta}, \boldsymbol{P}(\boldsymbol{\Theta})]$.[8] Suppose that there is no competitive equilibrium. Then by substitutability, there is a contract $\sigma^s = (\psi^s, p(\psi^s))$ such that $a(\sigma^s) = \{w, f\}$, and $\psi^s \in A^s - \Theta^s$; when f adds σ^s to its most favorable plan, it can be strictly better off, and when w adds σ^s to his most favorable plan, he will be indifferent. Otherwise there is a payment $p'(\psi^s)$ such that if w and f add $(\psi^s, p'(\psi^s))$ to their most favorable plan, neither of them can be strictly better off. Then, we can reset the payment of σ^s to $p'(\psi^s)$ and construct a competitive equilibrium. Let the set of these contracts (the contracts like σ^s) be Z^s in period s, and $\boldsymbol{Z} = (Z^1, ..., Z^T)$. Note that $\boldsymbol{Z}$ is finite.

Let $\boldsymbol{Y} = ([\Theta^1, P(\Theta^1)], ..., [\Theta^s, P(\Theta^s)] \cup \{\sigma^s\}, ..., [\Theta^T, P(\Theta^T)])$.

Define

$$D(\sigma^s, [\boldsymbol{\Theta}, \boldsymbol{P}(\boldsymbol{\Theta})]) \equiv \max_{\boldsymbol{Y}'_f \subseteq \boldsymbol{Y}_f} \{U_f(\boldsymbol{Y}'_f) - U_f([\boldsymbol{\Theta}_f, \boldsymbol{P}(\boldsymbol{\Theta}_f)])\}.$$

In words, $D(\cdot)$ is the (possibly negative) total possible gain realizeable by the firm f by adding the job $\psi^s = \alpha(\sigma^s)$ and upsetting the outcome $[\boldsymbol{\Theta}, \boldsymbol{P}(\boldsymbol{\Theta})]$. Define

$$F([\boldsymbol{\Theta}, \boldsymbol{P}(\boldsymbol{\Theta})]) \equiv \min_{\sigma^s \in \boldsymbol{Z}} D(\sigma^s, [\boldsymbol{\Theta}, \boldsymbol{P}(\boldsymbol{\Theta})]).$$

By hypothesis, $F([\boldsymbol{\Theta}, \boldsymbol{P}(\boldsymbol{\Theta})]) > 0$ as long as $[\boldsymbol{\Theta}, \boldsymbol{P}(\boldsymbol{\Theta})]$ is individually rational, since otherwise $\boldsymbol{Z} = \emptyset$ and we can construct a competitive equilibrium in the continuous market. We shall now show that $F(\cdot)$ is bounded above zero for all individually rational outcomes.

To see this, note that $F(\cdot)$ is continuous in $[\boldsymbol{\Theta}, \boldsymbol{P}(\boldsymbol{\Theta})]$ for any given $\boldsymbol{\Theta}$ because the maximum and minimum of continuous functions are continuous, and let

$$G(\boldsymbol{\Theta}) \equiv \min_{\boldsymbol{P}(\boldsymbol{\Theta})} F([\boldsymbol{\Theta}, \boldsymbol{P}(\boldsymbol{\Theta})]),$$

[8] Here $[\boldsymbol{\Theta}, \boldsymbol{P}(\boldsymbol{\Theta})]$ is a feasible set of contracts, and $\boldsymbol{P}(\boldsymbol{\Theta})$ specifies a payment for each job involved in $\boldsymbol{\Theta}$.

subject to
$\boldsymbol{P}(\boldsymbol{\Theta}) \geq \boldsymbol{P}(\boldsymbol{\Theta}(0))$, and $U_f([\boldsymbol{\Theta}_f, \boldsymbol{P}(\boldsymbol{\Theta}_f)]) \geq U_f([\boldsymbol{\Theta}'_f, \boldsymbol{P}(\boldsymbol{\Theta}'_f)])$ for all $f \in F$ and $\boldsymbol{\Theta}'_f \subseteq \boldsymbol{\Theta}_f$, where $\boldsymbol{P}(\boldsymbol{\Theta}(0))$ is the set of all permitted payments of jobs in $\boldsymbol{\Theta}$.

Then define

$$H \equiv \min_{\boldsymbol{\Theta} \subseteq \boldsymbol{A}} G(\boldsymbol{\Theta}).$$

$G(\cdot)$ is well-defined because for any given $\boldsymbol{\Theta} \subseteq \boldsymbol{A}$, F is continuous and the feasible region of the problem on the right-hand side of $G(\boldsymbol{\Theta}) \equiv \min_{\boldsymbol{P}(\boldsymbol{\Theta})} F([\boldsymbol{\Theta}, \boldsymbol{P}(\boldsymbol{\Theta})])$ is nonempty and compact. Further, $G(\boldsymbol{\Theta}) > 0$ for all $\boldsymbol{\Theta} \subseteq \boldsymbol{A}$ because, as noted above, $F(\cdot) > 0$ everywhere in the feasible region for all $\boldsymbol{\Theta} \subseteq \boldsymbol{A}$. Finally, H is well-defined and strictly positive because $\boldsymbol{A}$ is a finite set. Thus choosing the unit of measurement $\varepsilon < \frac{H}{(m+n)(T+1)}$ will suffice for the validity of the above arguments.

Thus from the above lemmas, the proof of the theorem is completed.

Proof of Theorem 2: Since each agent $i \in W \cup F$ is maximizing his utility in the competitive equilibrium $[\boldsymbol{\Theta}, \boldsymbol{P}]$, for any $\boldsymbol{\Theta}' \subseteq \boldsymbol{A}$, we have

$$\begin{aligned}
&\sum_{s=1}^{T} \sum_{i \in W \cup F} \delta^{s-1} v_i^s(\Theta_i^s) + \sum_{s=1}^{T} \sum_{i \in W} \delta^{s-1} p_i^s(\Theta_i^s) - \sum_{s=1}^{T} \sum_{i \in F} \delta^{s-1} p_i^s(\Theta_i^s) \\
&= \sum_{s=1}^{T} \sum_{i \in W \cup F} \delta^{s-1} v_i^s(\Theta_i^s) = \sum_{s=1}^{T} \sum_{i \in W \cup F} \delta^{s-1} u_i^s(\kappa[\Theta^s, P^s]) \\
&= \sum_{i \in W \cup F} U_i(\kappa[\boldsymbol{\Theta}, \boldsymbol{P}]) \geq \sum_{i \in W \cup F} U_i(\kappa[\boldsymbol{\Theta}', \boldsymbol{P}]) \\
&= \sum_{s=1}^{T} \sum_{i \in W \cup F} \delta^{s-1} v_i^s(\Theta_i'^s) + \sum_{s=1}^{T} \sum_{i \in W} \delta^{s-1} p_i^s(\Theta_i'^s) - \sum_{s=1}^{T} \sum_{i \in F} \delta^{s-1} p_i^s(\Theta_i'^s) \\
&= \sum_{s=1}^{T} \sum_{i \in W \cup F} \delta^{s-1} v_i^s(\Theta_i'^s) = \sum_{s=1}^{T} \sum_{i \in W \cup F} \delta^{s-1} u_i^s(\kappa[\Theta'^s, P^s]),
\end{aligned}$$

where $p_i^s(\Theta_i^s)$ and $p_i^s(\Theta_i'^s)$ are the sum of agent i's payment associated to the set of jobs Θ_i^s and $\Theta_i'^s$ in the arrangement $[\boldsymbol{\Theta}, \boldsymbol{P}]$ and $[\boldsymbol{\Theta}', \boldsymbol{P}]$, respectively. Therefore, $\boldsymbol{\Theta}$ is an efficient job plan, and for any $\boldsymbol{\Theta}' \subseteq \boldsymbol{A}$, we have

$$\sum_{s=1}^{T} \sum_{i \in W \cup F} \delta^{s-1} v_i^s(\Theta_i^s) \geq \sum_{s=1}^{T} \sum_{i \in W \cup F} \delta^{s-1} v_i^s(\Theta_i'^s).$$

Theorem 2 is proved.

Proof of Theorem 3: Since $\boldsymbol{\Psi} \subseteq \boldsymbol{A}$ is an efficient job plan, we have

$$\begin{aligned}
&\sum_{s=1}^{T}\sum_{i\in W\cup F}\delta^{s-1}v_i^s(\Psi_i^s)+\sum_{s=1}^{T}\sum_{i\in W}\delta^{s-1}p_i^s(\Psi_i^s)-\sum_{s=1}^{T}\sum_{i\in F}\delta^{s-1}p_i^s(\Psi_i^s)\\
&=\sum_{s=1}^{T}\sum_{i\in W\cup F}\delta^{s-1}v_i^s(\Psi_i^s)=\sum_{s=1}^{T}\sum_{i\in W\cup F}\delta^{s-1}u_i^s(\kappa([\Psi^s,P^s]))=\sum_{i\in W\cup F}U_i(\kappa[\boldsymbol{\Psi},\boldsymbol{P}])\\
&\geq\sum_{s=1}^{T}\sum_{i\in W\cup F}\delta^{s-1}v_i^s(\Theta_i^s)+\sum_{s=1}^{T}\sum_{i\in W}\delta^{s-1}p_i^s(\Theta_i^s)-\sum_{s=1}^{T}\sum_{i\in F}\delta^{s-1}p_i^s(\Theta_i^s)\\
&=\sum_{s=1}^{T}\sum_{i\in W\cup F}\delta^{s-1}v_i^s(\Theta_i^s)=\sum_{s=1}^{T}\sum_{i\in W\cup F}\delta^{s-1}u_i^s(\kappa([\Theta^s,P^s]))=\sum_{i\in W\cup F}U_i(\kappa[\boldsymbol{\Theta},\boldsymbol{P}]),
\end{aligned}$$

where $p_i^s(\Psi_i^s)$ and $p_i^s(\Theta_i^s)$ are the sum of agent i's payment associated to the set of jobs Ψ_i^s and Θ_i^s in the arrangement $[\boldsymbol{\Psi},\boldsymbol{P}]$ and $[\boldsymbol{\Theta},\boldsymbol{P}]$, respectively. Also, since $[\boldsymbol{\Theta},\boldsymbol{P}]$ is a competitive equilibrium, we have for each $i \in W \cup F$:

$$U_i(\kappa[\boldsymbol{\Theta},\boldsymbol{P}]) \geq U_i(\kappa[\boldsymbol{\Psi},\boldsymbol{P}]).$$

Thus, by substitutability, the inequality

$$\sum_{i\in W\cup F}U_i(\kappa[\boldsymbol{\Psi},\boldsymbol{P}]) \geq \sum_{i\in W\cup F}U_i(\kappa[\boldsymbol{\Theta},\boldsymbol{P}])$$

can hold only if for each $i \in W \cup F$, $U_i(\kappa[\boldsymbol{\Theta},\boldsymbol{P}]) = U_i(\kappa[\boldsymbol{\Psi},\boldsymbol{P}])$. Therefore, for each $i \in W \cup F$, we have $\boldsymbol{\Psi}_i \in D_i(\boldsymbol{P})$. Thus, $[\boldsymbol{\Psi},\boldsymbol{P}]$ is a competitive equilibrium.

Theorem 3 is proved.

Proof of Theorem 4: Given a payment plan $\boldsymbol{P}$, set $V_i(\boldsymbol{P}) \equiv \max_{\boldsymbol{\Theta}\subseteq\boldsymbol{A}} U_i(\kappa[\boldsymbol{\Theta},\boldsymbol{P}])$ and $V(\boldsymbol{P}) \equiv \sum_{i\in W\cup F} V_i(\boldsymbol{P})$. Let $\boldsymbol{\Theta}^* \subseteq \boldsymbol{A}$ be an efficient job plan, and

$$U^* = \sum_{i\in W\cup F} U_i(\kappa[\boldsymbol{\Theta}^*,\boldsymbol{P}]).$$

We first show a lemma that is similar to Sun and Yang (2009).

Lemma 4 *A payment plan $\boldsymbol{P}'$ is a competitive equilibrium payment plan if and only if:*

$$\boldsymbol{P}' \in \arg\min_{\boldsymbol{P}} V(\boldsymbol{P}).$$

Proof: To prove the first implication of the lemma, let $\boldsymbol{P}'$ be a competitive equilibrium payment plan and $\boldsymbol{P}$ an arbitrary payment plan. For each agent $i \in W \cup F$, consider some arbitrary $\boldsymbol{\Theta}^i \in D_i(\boldsymbol{P})$. By construction and utility maximization, we have

$$\begin{aligned} V(\boldsymbol{P}) = \sum_{i \in W \cup F} V_i(\boldsymbol{P}) = \sum_{i \in W \cup F} U_i(\kappa[\boldsymbol{\Theta}^i, \boldsymbol{P}]) \\ \geq \sum_{i \in W \cup F} U_i(\kappa[\boldsymbol{\Theta}^*, \boldsymbol{P}]) = U^* = V(\boldsymbol{P}'), \end{aligned}$$

where the last equality sign can be derived by the proof of Theorem 3.

Thus we have

$$\boldsymbol{P}' \in \arg\min_{\boldsymbol{P}} V(\boldsymbol{P}).$$

Now we show the other implication of this lemma. Suppose that $\boldsymbol{P}'$ is any payment plan that minimizes V (and thus satisfies $V(\boldsymbol{P}') = U^*$). By the definition of V_i, we have:

$$V_i(\boldsymbol{P}') \geq U_i(\kappa[\boldsymbol{\Theta}^*, \boldsymbol{P}']).$$

Summing the above inequality across $i \in W \cup F$ gives:

$$V(\boldsymbol{P}') = \sum_{i \in W \cup F} V_i(\boldsymbol{P}') \geq \sum_{i \in W \cup F} U_i(\kappa[\boldsymbol{\Theta}^*, \boldsymbol{P}']) = U^*.$$

If $V_i(\boldsymbol{P}') > U_i(\kappa[\boldsymbol{\Theta}^*, \boldsymbol{P}'])$ for some $i \in W \cup F$, we have $V(\boldsymbol{P}') > U^*$, which contradicts the hypothesis that $\boldsymbol{P}'$ minimizes V. Thus we have $V(\boldsymbol{P}') = U^*$. Therefore, for all $i \in W \cup F$, $V_i(\boldsymbol{P}') = U_i(\kappa[\boldsymbol{\Theta}^*, \boldsymbol{P}'])$, and thus $[\boldsymbol{\Theta}^*, \boldsymbol{P}']$ is a competitive equilibrium. The lemma has been proved.

Now, suppose that $\boldsymbol{P}$ and $\boldsymbol{Q}$ are two competitive equilibrium payment plans, and let $\boldsymbol{P} \wedge \boldsymbol{Q}$ and $\boldsymbol{P} \vee \boldsymbol{Q}$ be their meet and join, respectively. Thus, by Lemma 4 and the submodularity of V (since agents' preferences satisfy substitutability, for each $i \in W \cup F$, V_i is submodular [9]), we have:

$$\begin{aligned} 2U^* \leq V(\boldsymbol{P} \wedge \boldsymbol{Q}) + V(\boldsymbol{P} \vee \boldsymbol{Q}) \\ \leq V(\boldsymbol{P}) + V(\boldsymbol{Q}) = 2U^*. \end{aligned}$$

Since $V(\boldsymbol{P} \wedge \boldsymbol{Q}) \geq U^*$ and $V(\boldsymbol{P} \vee \boldsymbol{Q}) \geq U^*$, we have $V(\boldsymbol{P} \wedge \boldsymbol{Q}) = V(\boldsymbol{P} \vee \boldsymbol{Q}) = U^*$. Then by Lemma 4, $\boldsymbol{P} \wedge \boldsymbol{Q}$ and $\boldsymbol{P} \vee \boldsymbol{Q}$ are competitive equilibrium payment plans.

[9] A similar result has been proved in Sun and Yang (2009) (Theorem 2).

The proof of Theorem 4 is completed.

Proof of Theorem 5: We first show the following lemma.

Lemma 5 *If* $[\boldsymbol{\Theta},\boldsymbol{P}]$ *is a competitive equilibrium, then for any* $t = 1,...,T$, $[\boldsymbol{\Theta}^{\geq t},\boldsymbol{P}^{\geq t}]$ *is a competitive equilibrium in the continuation market at* $h^t = (\emptyset,\kappa[\boldsymbol{\Theta}^1,\boldsymbol{P}^1],...,\kappa[\boldsymbol{\Theta}^{t-1},\boldsymbol{P}^{t-1}])$.

Proof. Suppose by way of contradiction that from period t, $[\boldsymbol{\Theta}^{\geq t},\boldsymbol{P}^{\geq t}]$ is not a competitive equilibrium. Then there is some $i \in W \cup F$ such that $\boldsymbol{\Theta}_i^{\geq t} \notin D_i(\boldsymbol{P}^{\geq t})$. This implies given $\boldsymbol{P}^{\geq t}$ and $h^t = (\emptyset,\kappa[\boldsymbol{\Theta}^1,\boldsymbol{P}^1],...,\kappa[\boldsymbol{\Theta}^{t-1},\boldsymbol{P}^{t-1}])$, agent i is not optimizing from period t. Then, there is a $\boldsymbol{\Theta}' = (\Theta^1,...,\Theta^{t-1},\Theta'^t,...,\Theta'^T)$ such that $U_i(\kappa[\boldsymbol{\Theta}',\boldsymbol{P}]) > U_i(\kappa[\boldsymbol{\Theta},\boldsymbol{P}])$. Thus, given $\boldsymbol{P}$, agent i is not optimizing at the outcome $\kappa[\boldsymbol{\Theta},\boldsymbol{P}]$, contradicting the hypothesis that $[\boldsymbol{\Theta},\boldsymbol{P}]$ is a competitive equilibrium.

The lemma is proved.

By Theorem 1, if agents' preferences satisfy substitutability, then there exists a competitive equilibrium $[\boldsymbol{\Theta},\boldsymbol{P}]$. Suppose that the outcome $\boldsymbol{\Omega} = \kappa[\boldsymbol{\Theta},\boldsymbol{P}]$ is induced by the competitive equilibrium $[\boldsymbol{\Theta},\boldsymbol{P}]$, then by Theorem 2, it is efficient. Now we show that it is in the IRRC.

Suppose by way of contradiction that $\boldsymbol{\Omega} = \kappa[\boldsymbol{\Theta},\boldsymbol{P}]$ is not in the IRRC, then there exists a group $S \subseteq W \cup F$ with a plan $\boldsymbol{\Omega}_S'^{\geq t} \neq \boldsymbol{\Omega}_S^{\geq t}$ that can block the outcome from some period t.

Suppose that there exists a group $S \subseteq W \cup F$ with a plan $\boldsymbol{\Omega}_S'^{\geq t} \neq \boldsymbol{\Omega}_S^{\geq t}$, such that $U_i^t(\boldsymbol{\Omega}_{iS}'^{\geq t}) \geq U_i^t(\boldsymbol{\Omega}_{iS}^{\geq t})$ for all $i \in S$, and $U_j^t(\boldsymbol{\Omega}_{jS}'^{\geq t}) > U_j^t(\boldsymbol{\Omega}_{jS}^{\geq t})$ for at least $j \in S$. Then we have:

$$\sum_{i\in S} U_i^t(\boldsymbol{\Omega}_{iS}'^{\geq t}) > \sum_{i\in S} U_i^t(\boldsymbol{\Omega}_{iS}^{\geq t}).$$

Since $[\boldsymbol{\Theta},\boldsymbol{P}]$ is competitive equilibrium, by Lemma 5, $\alpha(\boldsymbol{\Omega}_{iS}^{\geq t}) \in D_i(\boldsymbol{P}^{\geq t})$ for all $i \in W \cup F$ and $t = 1,...,T$.

Thus, $U_i^t(\kappa[\alpha(\boldsymbol{\Omega}_{iS}^{\geq t}),\boldsymbol{P}^{\geq t}]) \geq U_i^t(\kappa[\alpha(\boldsymbol{\Omega}_{iS}'^{\geq t}),\boldsymbol{P}^{\geq t}])$ for all $i \in S$ and $t \in \{1,\ldots,T\}$. Then,

$$\sum_{i\in S} U_i^t(\kappa[\alpha(\boldsymbol{\Omega}_{iS}^{\geq t}),\boldsymbol{P}^{\geq t}]) = \sum_{i\in S} U_i^t(\boldsymbol{\Omega}_{iS}^{\geq t})$$
$$\geq \sum_{i\in S} U_i^t(\kappa[\alpha(\boldsymbol{\Omega}_{iS}'^{\geq t}),\boldsymbol{P}^{\geq t}]) = \sum_{i\in S} U_i^t(\boldsymbol{\Omega}_{iS}'^{\geq t}),$$

a contradiction. Thus, $\boldsymbol{\Omega} = \kappa[\boldsymbol{\Theta},\boldsymbol{P}]$ is in the IRRC.

Summarizing the above discussion, the proof is completed.

Proof of Theorem 6: Suppose by way of contradiction that there is an IRRC outcome $\mathbf{\Omega} = [\mathbf{\Psi}, \boldsymbol{P}(\mathbf{\Psi})]$ that is not efficient. Then there is a $\mathbf{\Omega}' \neq \mathbf{\Omega}$ such that

$$\sum_{i \in W \cup F} U_i(\mathbf{\Omega}') > \sum_{i \in W \cup F} U_i(\mathbf{\Omega}).$$

Let $\alpha(\mathbf{\Omega}') = \mathbf{\Psi}'$, then

$$\begin{aligned}
&\sum_{i \in W \cup F} U_i(\mathbf{\Omega}') = \sum_{s=1}^{T} \sum_{i \in W \cup F} \delta^{s-1} v_i^s(\Psi_i'^s) + \sum_{s=1}^{T} \sum_{i \in W} \delta^{s-1} p_i^s(\Psi_i'^s) - \sum_{s=1}^{T} \sum_{i \in F} \delta^{s-1} p_i^s(\Psi_i'^s) \\
&= \sum_{s=1}^{T} \sum_{i \in W \cup F} \delta^{s-1} v_i^s(\Psi_i'^s) > \sum_{i \in W \cup F} U_i(\mathbf{\Omega}) \\
&= \sum_{s=1}^{T} \sum_{i \in W \cup F} \delta^{s-1} v_i^s(\Psi_i^s) + \sum_{s=1}^{T} \sum_{i \in W} \delta^{s-1} p_i^s(\Psi_i^s) - \sum_{s=1}^{T} \sum_{i \in F} \delta^{s-1} p_i^s(\Psi_i^s) \\
&= \sum_{s=1}^{T} \sum_{i \in W \cup F} \delta^{s-1} v_i^s(\Psi_i^s),
\end{aligned}$$

where $p_i^s(\Psi_i'^s)$ and $p_i^s(\Psi_i^s)$ are the sum of agent i's payment associated to the set of jobs $\Psi_i'^s$ and Ψ_i^s in the outcome $\mathbf{\Omega}'$ and $\mathbf{\Omega}$, respectively.

Thus, we have:

$$\sum_{s=1}^{T} \sum_{i \in W \cup F} \delta^{s-1} v_i^s(\Psi_i'^s) > \sum_{s=1}^{T} \sum_{i \in W \cup F} \delta^{s-1} v_i^s(\Psi_i^s).$$

By construction, let $(S_1, \ldots, S_K)$ be a partition of $W \cup F$. Then, for any $i \neq j$, $S_i \cap S_j = \emptyset$, and $\bigcup_{k=1}^{K} S_k = W \cup F$. Moreover, for any $w \in W \cap S_i$, $f \in F \cap S_i$ and S_i, let $\mathbf{\Psi}'_w \cap \mathbf{\Psi}'_f \cap \mathbf{\Psi}'_{S_i} \neq \emptyset$. Then, we can make transfers between any two agents in any group S_i at the outcome $\mathbf{\Omega}'$.

Since $\sum_{s=1}^{T} \sum_{i \in W \cup F} \delta^{s-1} v_i^s(\Psi_i'^s) > \sum_{s=1}^{T} \sum_{i \in W \cup F} \delta^{s-1} v_i^s(\Psi_i^s)$, there is at least one group S_k such that

$$\sum_{s=1}^{T} \sum_{i \in S_k} \delta^{s-1} v_i^s(\Psi_i'^s) > \sum_{s=1}^{T} \sum_{i \in S_k} \delta^{s-1} v_i^s(\Psi_i^s).$$

Note that for any $\boldsymbol{P}$, we have

$$\begin{aligned}\sum_{i\in S_k} U_i(\kappa[\boldsymbol{\Psi}',\boldsymbol{P}]) &= \sum_{i\in S_k}\delta^{s-1}u_i^s(\kappa[\Psi'^s,P^s]) \\ &= \sum_{s=1}^{T}\sum_{i\in S_k}\delta^{s-1}v_i^s(\Psi_i'^s) > \sum_{s=1}^{T}\sum_{i\in S_k}\delta^{s-1}v_i^s(\Psi_i^s) \\ &= \sum_{i\in S_k}\delta^{s-1}u_i^s(\kappa[\Psi^s,P^s]) = \sum_{i\in S_k}U_i(\kappa[\boldsymbol{\Psi},\boldsymbol{P}]).\end{aligned}$$

Since we can make transfers between any two agents in group S_k at the outcome $\boldsymbol{\Omega}'$, there must exist a $\boldsymbol{P}(\boldsymbol{\Psi}')$ such that $\boldsymbol{\Omega}'' = [\boldsymbol{\Psi}',\boldsymbol{P}(\boldsymbol{\Psi}')]$ and $U_i(\boldsymbol{\Omega}'') = \sum_{s=1}^{T}\delta^{s-1}u_i^s(\Omega_i''^s) > \sum_{s=1}^{T}\delta^{s-1}u_i^s(\Omega_i^s) = U_i(\boldsymbol{\Omega})$ for any $i\in S_k$. This implies that the group S_k with the plan $\boldsymbol{\Omega}''_{S_k}$ can block the outcome $\boldsymbol{\Omega}$ from period 1, contradicting the hypothesis that $\boldsymbol{\Omega} = [\boldsymbol{\Psi},\boldsymbol{P}(\boldsymbol{\Psi})]$ is in the IRRC. Thus, any IRRC outcome is efficient.

Summarizing the above discussion, the proof is completed.

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