Mid term exam 1 (Notes, books, and calculators are not authorized)

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with no justification will not be graded.

**Question 1:** Find a parametric representation of the line in $\mathbb{R}^4$ that passes through $P = (1, 2, 3, 4)$ and is orthogonal to the hyperplane $x_1 - 2x_2 + 3x_3 - 4x_4 = 5$.

A normal vector to the hyperplane $x_1 - 2x_2 + 3x_3 - 4x_4 = 5$ is $u := (1, -2, 3, -4)$. The line in question is the collection of points $X = (x_1, x_2, x_3, x_4)$ so that $X - P$ is parallel to $u$. This means that there is a real number $t$ so that

$$X - P = tu = (t, -2t, 3t, -4t).$$

The parametric representation of the line is

$$x_1 = 1 + t, \quad x_2 = 2 - 2t, \quad x_3 = 3 + 3t, \quad x_4 = 4 - 4t.$$

**Question 2:** Find a unit vector orthogonal to $u = (1, 2, 3)$ and $v = (3, 2, 1)$.

It is a known property of the cross product that the following vector $w := u \times v$ is orthogonal to $u$ and $v$. By definition

$$w = \begin{vmatrix} 2 & 2 & 1 \\ 3 & 1 & 2 \\ 1 & 3 & 2 \end{vmatrix} = (-4, 8, -4).$$

By normalizing $w$ we obtain the desired vector:

$$\frac{w}{\|w\|} = \frac{1}{\sqrt{96}} (-4, 8, -4).$$
**Question 3:** Find an equation of the hyperplane in $\mathbb{R}^4$ that passes through $P = (0, 1, 2, 3)$ and is normal to $u = (-1, -1, 1, 1)$.

By definition $H$ is the collection of points $X = (x_1, x_2, x_3, x_4)$ so that $(X - P) \cdot u$. This means

$$0 = (X - P) \cdot (-1, -1, 1, 1) = (x_1, x_2 - 1, x_3 - 2, x_4 - 3) \cdot (-1, -1, 1, 1)$$

$$= -x_1 - (x_2 - 1) + (x_3 - 2) + (x_4 - 3) = -x_1 - x_2 + x_3 + x_4 - 4.$$ 

The equation of the hyperplane in question

$$0 = -x_1 - x_2 + x_3 + x_4 - 4.$$

**Question 4:** Prove the triangle inequality: $\|u + v\| \leq \|u\| + \|v\|$ for all $u, v \in \mathbb{R}^n$.

By definition of the dot product, we have the following equality:

$$\|u + v\|^2 := (u + v)^2 = (u + v) \cdot (u + v) = u \cdot u + u \cdot v + v \cdot u + v \cdot v.$$

Using the Cauchy-Schwarz inequality, we obtain

$$\|u + v\|^2 \leq \|u\|^2 + \|u\| \|v\| + \|v\| \|u\| + \|v\|^2 = (\|u\| + \|v\|)^2.$$

Taking the square root gives the triangle inequality:

$$\|u + v\| \leq \|u\| + \|v\|.$$
Question 5: Find the inverse of \( A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \)

The determinant of \( A \) is \( |A| = 4 - 16 = -2 \neq 0 \). The matrix is invertible. We use the formula from class to compute the inverse:

\[ A^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \]

Question 6: Let \( A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \) and \( B = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \). Find (a) \( 3A - 4B \) (b) \( A^T \) (c) \( A^T B \).

\[
3A - 4B = \begin{bmatrix} 3 & -3 & 3 \\ -3 & 3 & -3 \end{bmatrix} - 4 \begin{bmatrix} 4 & -8 & -12 \\ -12 & -8 & -4 \end{bmatrix} = \begin{bmatrix} -1 & -11 & -9 \\ -15 & -5 & -7 \end{bmatrix}
\]

\[ A^T = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix} \]

\[ A^T B = \begin{bmatrix} -2 & 0 & 2 \\ 2 & 0 & -2 \\ -2 & 0 & 2 \end{bmatrix} \]
Question 7: Let $A$ be a square matrix. Assume that $A$ has the following property: $X^TAX \geq 2\|X\|^2$ for all column vector $X$. Prove that $A$ is invertible.

Consider $X$ so that $AX = 0$. Then $0 = X^TAX \geq 2\|X\|^2$. This proves that $\|X\| = 0$, which in turn proves that $X = 0$. As a result the solution set of $AX = 0$ is $\{0\}$. This proves that $A$ is invertible since $A$ is square.

Question 8: Find $2 \times 2$ nonzero matrices $A$ and $B$ such that $AB = 0$.

Consider the following matrix:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = A$$

Then

$$AB = 0$$
Question 9: Let \( A = \begin{bmatrix} 1 & 2 & -3 \\ -3 & -4 & 13 \\ 2 & 1 & -5 \end{bmatrix} \). Find the LU factorization of \( A \).

We compute the echelon form of \( A \) to get \( U \) and \( L \):

\[
A \begin{bmatrix} 1 & 2 & -3 \\ -3 & -4 & 13 \\ 2 & 1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 2 & 4 \\ 0 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 2 & 4 \\ 0 & 0 & 7 \end{bmatrix}
\]

\[
l_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \quad l_2 = \begin{bmatrix} 0 \\ 1 \\ -\frac{3}{2} \end{bmatrix}, \quad l_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]

Finally we have

\[
L = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & -\frac{3}{2} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 2 & 4 \\ 0 & 0 & 7 \end{bmatrix}
\]
Question 10: Let \( u_1 = (1, 2, 4), u_2 = (2, -3, 1), u_3 = (2, 1, -1) \) in \( \mathbb{R}^3 \). (a) Show that \( u_1, u_2, u_3 \) are orthogonal.

We compute \( u_i \cdot u_j \) for \( i \neq j, i, j \in \{1, 2, 3\} \).

\[
\begin{align*}
    u_1 \cdot u_2 &= 2 - 6 + 4 = 0, \\
    u_1 \cdot u_3 &= 2 + 2 - 4 = 0, \\
    u_2 \cdot u_3 &= 4 - 3 - 1 = 0.
\end{align*}
\]

This proves the statement.

(b) consider the \( 3 \times 3 \) matrix \( U = [u_1 u_2 u_3] \) where \( u_j, j \in \{1, 2, 3\} \) are considered as column vectors. Show that \( U \) is invertible.

Consider \( X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \) so that \( UX = 0 \). Then

\[
x_1 u_1 + x_2 u_2 + x_3 u_3 = 0.
\]

Taking the dot product with \( u_1 \) gives

\[
0 = x_1 u_1 \cdot u_1 + x_2 u_2 \cdot u_1 + x_3 u_3 \cdot u_1 = x_1 \|u_1\|^2 = 21x_1.
\]

This proves that \( x_1 = 0 \). By repeating this process with \( u_2 \) and \( u_3 \) we obtain that \( x_2 = 0 \) and \( x_3 = 0 \). As a result the solution set of \( UX = 0 \) is \( \{0\} \). This means that \( U \) is invertible.

(c) Let \( v = (3, 5, 2) \). Solve \( UX = v \), where \( v \) is considered as column vector. (Do not compute the reduced echelon form of the system.)

By definition

\[
x_1 u_1 + x_2 u_2 + x_3 u_3 = v
\]

Taking the dot product with \( u_1 \) and using the orthogonality of \( u_1, u_2, u_3 \) gives

\[
v \cdot u_1 = 21 = x_1 u_1 \cdot u_1 + x_2 u_2 \cdot u_1 + x_3 u_3 \cdot u_1 = x_1 \|u_1\|^2 = 21x_1,
\]

i.e., \( x_1 = 1 \). Similarly

\[
v \cdot u_2 = -7 = x_1 u_1 \cdot u_2 + x_2 u_2 \cdot u_2 + x_3 u_3 \cdot u_2 = x_2 \|u_2\|^2 = 14x_2,
\]

i.e., \( x_1 = -\frac{1}{2} \).

\[
v \cdot u_3 = 9 = x_1 u_1 \cdot u_3 + x_2 u_2 \cdot u_3 + x_3 u_3 \cdot u_3 = x_3 \|u_3\|^2 = 6x_3,
\]

i.e., \( x_1 = \frac{3}{2} \). Hence the solution is

\[
X = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ \frac{3}{2} \end{bmatrix}.
\]