Mid term exam 2 (Notes, books, and calculators are not authorized). Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with no justification will not be graded. Some useful facts: $(f, g) \mapsto \int_a^b f(t)g(t)dt$ is an inner product in $C^0([a, b]; \mathbb{R})$ and $(A, B) \mapsto \text{tr}(A^TB)$ is an inner product for square matrices.

**Question 1:** Are the following polynomials $p_1(t) = 1 - 3t + 2t^2$, $p_2(t) = 2 - 4t - t^2$, $p_3(t) = 1 - 5t + 7t^2$ linearly independent in $P^2(\mathbb{R})$?

Let $X = (x_1, x_2, x_3) \in \mathbb{R}^3$ be so that $x_1p_1 + x_2p_2 + x_3p_3 = 0$. Then

$$x_1p_1(t) + x_2p_2(t) + x_3p_3(t) = x_1 + 2x_2 + x_3 + (-3x_1 - 4x_2 - 5x_3)t + (2x_1 - 2x_2 + 7x_3)t^2 = 0.$$  

This means that $X$ solves

$$\begin{bmatrix} 1 & 2 & 1 \\ -3 & -4 & -5 \\ 2 & -1 & 7 \end{bmatrix} X = 0.$$  

Let us reduce the matrix of the linear system in echelon form

$$\begin{bmatrix} 1 & 2 & 1 \\ -3 & -4 & -5 \\ 2 & -1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & -5 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$  

There are only two pivots. There is one free variable. This means that the solution set of the above linear system is not $\{0\}$. There is some nonzero vector $X$ so that $x_1p_1 + x_2p_2 + x_3p_3 = 0$. This means that $p_1, p_2, p_3$ are linearly dependent.

**Question 2:** Let $(V, \langle \cdot, \cdot \rangle)$ be a finite-dimensional inner product space. Let $F : V \rightarrow V$ be a linear mapping. Assume that $F$ has the following property: $\langle F(v), v \rangle \geq 2\|v\|^2$ for all vectors $v$ in $V$. Prove that $F$ is bijective.

Let us characterize $\text{Ker}(F)$. Consider $v \in \text{Ker}(F)$. Then $F(v) = 0$. This implies that $0 = \langle F(v), v \rangle \geq 2\|v\|^2$. This proves that $\|v\| = 0$, which in turn proves that $v = 0$. As a result $\text{Ker}(F) = \{0\}$. This proves that $F$ is injective. Owing to the rank theorem, this also proves that $F$ is surjective since the dimension of the co-domain is equal to the dimension of the domain. Hence $F$ is bijective.
Question 3: Is it true that $\text{tr}((A + B)^T(A + B))^{\frac{1}{2}} \leq \text{tr}(A^T A)^{\frac{1}{2}} + \text{tr}(B^T B)^{\frac{1}{2}}$ for all square matrices $A$ and $B$? If yes, why (be very specific); if no, give a counter-example.

The mapping $(A, B) \mapsto \text{tr}(A^T B)$ is an inner product. We have proved it in class. This inner product induces a norm: $\|A\| = \text{tr}(A^T A)^{\frac{1}{2}}$. The proposed formula is just the triangle inequality,

$$\|A + B\| \leq \|A\| + \|B\|,$$

which we know from class is true for all norms.

Question 4: Is it true that $\left| \int_{\text{log } \pi} e^{\log \pi} f(t)g(t)dt \right| \leq \left( \int_{\text{log } \pi} e^{\log \pi} f(t)^2 dt \right)^{\frac{1}{2}} \left( \int_{\text{log } \pi} e^{\log \pi} g(t)^2 dt \right)^{\frac{1}{2}}$ for all continuous functions? If yes, why (be very specific); if no, give a counter-example.

The mapping $(f, g) \mapsto \int_{\text{log } \pi} e^{\log \pi} f(t)g(t)dt$ is an inner product. We have proved it in class. This inner product induces a norm: $\|f\| = \left( \int_{\text{log } \pi} e^{\log \pi} f(t)^2 dt \right)^{\frac{1}{2}}$. The proposed formula is just the Cauchy-Schwarz inequality,

$$\langle f, g \rangle \leq \|f\| \|g\|,$$

which we know from class is true for all inner products.
**Question 5:** Find two nonzero linear mappings \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) and \( F : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) so that \( F(T(X)) = 0 \) for all \( X \) in \( \mathbb{R}^2 \).

Let \( A \) and \( B \) two matrices in \( M_{2,2}(\mathbb{R}) \) and consider the following mappings:

\[
T(X) = BX, \quad F(X) = AX.
\]

Then \( F(T(X)) = F(BX) = ABX \). The problem consists of finding \( A \neq 0 \) and \( B \neq 0 \) so that \( AB = 0 \). Let us consider the following matrices

\[
A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = A
\]

Then \( AB = 0 \).

In conclusion the mappings \( F \) and \( T \) defined above are not zero and are so that \( F(T(X)) = 0 \) for all \( X \) in \( \mathbb{R}^2 \).

**Question 6:** Let \( F : \mathbb{P}_1(t) \rightarrow \mathbb{P}_1(t) \) be the linear mapping such that \( F(t) = 1 + 4t \) and \( F(1+2t) = 2 + 3t \). Find \( F(a + bt) \) for all pairs \( (a, b) \) in \( \mathbb{R}^2 \).

Let \( a + bt \) in \( \mathbb{P}_1(t) \). We write \( a + bt \) as a linear combination of \( t \) and \( 1 + 2t \). (This is always possible since \( \{t, 1+2t\} \) is clearly a basis of \( \mathbb{P}_1(t) \).) Let us find the two real numbers \( x \) and \( y \) so that \( a + bt = xt + y(1 + 2t) \). This is equivalent to saying

\[
a = y, \quad \text{and} \quad b = x + 2y.
\]

This implies

\[
x = b - 2a \quad \text{and} \quad y = a.
\]

Then, the following holds owing to the fact that the mapping \( F \) is linear

\[
F(a + bt) = F(xt + y(1 + 2t)) =xF(t) + yF(1 + 2t) = (b - 2a)(1 + 4t) + a(2 + 3t) = b - 2a + 2a + (4b - 8a + 3a)t.
\]

In conclusion

\[
F(a + bt) = b + (4b - 5a)t.
\]
Question 7: Consider the set of two-variate polynomials of degree at most one \( P_1(s,t) = \{as + bt + c; (a,b,c) \in \mathbb{R}^3\} \). Consider \( F : P_1(s,t) \rightarrow \mathbb{R}^3 \) the linear mapping defined by \( F(p) = (p(0,0), p(1,1), p(4,2)) \). Show that \( F \) is bijective (Hint: observe that dim(\( P_1(s,t) \)) = 3).

Since \( P_1(s,t) \) and \( \mathbb{R}^3 \) have the same dimension, it suffices to prove that \( F \) is injective (by the rank theorem). Let \( p = c + bs + at \) be a member of \( P_1(s,t) \) such that \( F(p) = 0 \). Then

\[
0 = p(0,0) = c, \quad 0 = p(1,1) = a + b + c \quad 0 = p(4,2) = 4a + 2b + c.
\]

This is a linear system for \( a, b \) and \( c \). The associated matrix is

\[
\begin{bmatrix}
4 & 2 & 1 \\
1 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix} \sim \begin{bmatrix}
4 & 2 & 1 \\
0 & -2 & -3 \\
0 & 0 & 1
\end{bmatrix}
\]

The rank is maximum. This means that \( a = b = c = 0 \). As a result \( F \) is injective. In conclusion \( F \) is bijective.

Question 8: Let \( V = C^0([-\pi, \pi]; \mathbb{R}) \) be the vector space over \( \mathbb{R} \) composed of the functions from \([-\pi, \pi]\) to \( \mathbb{R} \) that are continuous. Consider the inner product \( \langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)g(t)dt \) and the associated norm \( \| \cdot \| \). We recall that \( \|1/\sqrt{2}\| = \|\sin(t)\| = \|\cos(t)\| = 1 \) and \( B = \{1/\sqrt{2}, \sin(t), \cos(t)\} \) are orthonormal. Let \( U = \text{span}(B) \). Let \( u_1(t) = 1/\sqrt{2} + 2\sin(t) + 4\cos(t) \), \( u_2(t) = 2/\sqrt{2} - 3\sin(t) + \cos(t) \), \( u_3(t) = 2/\sqrt{2} + \sin(t) - \cos(t) \) in \( V \). (a) Show that \( u_1, u_2, u_3 \) are orthogonal and compute \( \|u_1\|, \|u_2\|, \|u_3\| \).

We compute \( \langle u_i, u_j \rangle \) for \( i \neq j, i, j \in \{1, 2, 3\} \).

\[
\begin{align*}
\langle u_1, u_2 \rangle &= 2\|1/\sqrt{2}\| - 6\|\sin\| + 4\|\cos\| = 2 - 6 + 4 = 0, \\
\langle u_1, u_3 \rangle &= 2\|1/\sqrt{2}\| + 2\|\sin\| - 4\|\cos\| = 2 + 2 - 4 = 0, \\
\langle u_2, u_3 \rangle &= 4\|1/\sqrt{2}\| - 3\|\sin\| - 1\|\cos\| = 4 - 3 - 1 = 0.
\end{align*}
\]

This proves the statement. Clearly

\[
\|u_1\| = \sqrt{21}, \quad \|u_2\| = \sqrt{14}, \quad \|u_3\| = \sqrt{6}.
\]
(b) What is the dimension of \( U = \text{span}(B) \)? Is \( C = \{u_1(t), u_2(t), u_3(t)\} \) a basis of \( U \)? (why).

The vectors composing \( B \) are linearly independent since they are orthogonal. Moreover \( B \) is a spanning set of \( U \) by definition. Hence \( B \) is a basis of \( U \). As a result \( \dim(U) = 3 \).

\( C \) is a basis of \( U \) since (1) it is a subset of \( U \), (2) the vectors composing \( C \) are linearly independent (since they are orthogonal) and (3) the cardinal number of \( C \) equals the dimension of \( U \).

(c) Consider the linear mapping \( F : U \rightarrow U \) so that \( F(x/\sqrt{2} + y \sin(t) + z \cos(t)) = xu_1(t) + yu_2(t) + zu_3(t) \) for all \( x, y, z \in \mathbb{R}^3 \). Consider the base \( C = \{u_1(t), u_2(t), u_3(t)\} \). Compute the matrices \([F]_{B,B}\) and \([F]_{B,C}\)?

Observe that

\[
F(1/\sqrt{2}) = u_1(t) = 1/\sqrt{2} + 2\sin(t) + 4\cos(t),
\]

\[
F(\sin(t)) = u_2(t) = 2/\sqrt{2} - 3\sin(t) + \cos(t),
\]

\[
F(\cos(t)) = u_3(t) = 2/\sqrt{2} + \sin(t) - \cos(t)
\]

This means that

\[
[F]_{B,B} = \begin{bmatrix}
1 & 2 & 2 \\
2 & -3 & 1 \\
4 & 1 & -1
\end{bmatrix}
\]

and

\[
[F]_{B,C} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
(d) Prove that the linear mapping $F$ defined in question (c) is bijective.

There are many ways to prove this. The easiest answer is that $F$ is bijective because the matrix $[F]_B^C$ is clearly invertible. Another way to prove the statement consists of characterizing $\text{Ker}(F)$.

Let $v \in \text{Ker}(F)$. There are numbers $x, y, z$ in $\mathbb{R}^3$ so that $v = x/\sqrt{2} + y \sin(t) + z \cos(t)$ since $B$ is a basis of $U$. Then

$$F(v) = 0 = xu_1(t) + yu_2(t) + zu_3(t),$$

which implies

$$\langle F(v), u_1 \rangle = x\|u_1\|^2 = 0, \quad \langle F(v), u_2 \rangle = y\|u_2\|^2 = 0, \quad \langle F(v), u_3 \rangle = z\|u_3\|^2 = 0.$$

As a result $x = y = z = 0$, which in turn implies $v = 0$. This proves that $F$ is injective. Owing to the rank theorem, this also proves that $F$ is surjective since the dimension of the co-domain is equal to the dimension of the domain. Hence $F$ is bijective.

(e) Let $v(t) = 3/\sqrt{2} + 5 \sin(t) + 2 \cos(t)$. Find $u(t) \in U$ so that $F(u) = v$ (Hint: use the fact that $u_1, u_2, u_3$ are orthogonal).

Let $x_1, x_2, x_3 \in \mathbb{R}^3$ so that $u = x_1/\sqrt{2} + x_2 \sin(t) + x_3 \cos(t)$. By definition of $F$

$$v = F(u) = x_1 F(1/\sqrt{2}) + x_2 F(\sin(t)) + x_3 F(\cos(t)) = x_1 u_1 + x_2 u_2 + x_3 u_3$$

Taking the inner product with $u_1$ and using the orthogonality of $u_1, u_2, u_3$ gives

$$\langle v, u_1 \rangle = 21 = x_1 \langle u_1, u_1 \rangle + x_2 \langle u_2, u_1 \rangle + x_3 \langle u_3, u_1 \rangle = x_1 \|u_1\|^2 = 21 x_1,$$

i.e., $x_1 = 1$. Similarly

$$\langle v, u_2 \rangle = -7 = x_1 \langle u_1, u_2 \rangle + x_2 \langle u_2, u_2 \rangle + x_3 \langle u_3, u_2 \rangle = x_2 \|u_2\|^2 = 14 x_2,$$

i.e., $x_1 = -\frac{1}{2}$. Similarly

$$\langle v, u_3 \rangle = 9 = x_1 \langle u_1, u_3 \rangle + x_2 \langle u_2, u_3 \rangle + x_3 \langle u_3, u_3 \rangle = x_3 \|u_3\|^2 = 6 x_3,$$

i.e., $x_1 = \frac{3}{2}$. Hence the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ \frac{3}{2} \end{bmatrix}.$$

In other words

$$v(t) = 1/\sqrt{2} - \frac{1}{2} \sin(t) + \frac{3}{2} \cos(t)$$