Quizz 4 (Notes, books, and calculators are not authorized)

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with no justification will not be graded.

**Question 1:** Are \( u_1 = (1, -3, 2), u_2 = (2, -4, -1), u_3 = (1, -5, 7) \) linearly independent in \( \mathbb{R}^3 \)?

Let \( X = (x_1, x_2, x_3) \in \mathbb{R}^3 \) be so that \( x_1 u_1 + x_2 u_2 + x_3 u_3 = 0 \). Then \( X \) solves

\[
\begin{bmatrix}
1 & 2 & 1 \\
-3 & -4 & -5 \\
2 & -1 & 7
\end{bmatrix} X = 0.
\]

Let us reduce the matrix of the linear system in echelon form

\[
\begin{bmatrix}
1 & 2 & 1 \\
-3 & -4 & -5 \\
2 & -1 & 7
\end{bmatrix} \sim \begin{bmatrix}
1 & 2 & 1 \\
0 & 2 & -2 \\
0 & -5 & 5
\end{bmatrix} \sim \begin{bmatrix}
1 & 2 & 1 \\
0 & 2 & -2 \\
0 & 0 & 0
\end{bmatrix}.
\]

There are only two pivots. There is one free variable. This means that the solution set of the above linear system is not \( \{0\} \). There is some nonzero vector \( X \) so that \( x_1 u_1 + x_2 u_2 + x_3 u_3 = 0 \). This means that \( u_1, u_2, u_3 \) are linearly dependent.

**Question 2:** Express the polynomial \( v = t^2 + 4t - 3 \) as a linear combination of \( p_1 = t^2 - 2t + 5, p_2 = 2t^2 - 3t, p_3 = t + 3 \).

We look for \( (x_1, x_2, x_3) \in \mathbb{R}^3 \) so that \( v = x_1 p_1 + x_2 p_2 + x_3 p_3 \). For all \( t \in \mathbb{R} \) the following holds

\[
t^2 + 4t - 3 = x_1 (t^2 - 2t + 5) + x_2 (2t^2 - 3t) + x_3 (t + 3) = t^2(x_1 + 2x_2) + t(-2x_1 - 3x_2 + x_3) + (5x_1 + 3x_3).
\]

This implies \( x_1 + 2x_2 = 1, \quad -2x_1 - 3x_2 + x_3 = 4, \quad 5x_1 + 3x_3 = -3 \).

We can write this system in the form of an augmented matrix and compute the reduced echelon form:

\[
\begin{bmatrix}
1 & 2 & 0 & 1 \\
-2 & -3 & 1 & 4 \\
5 & 0 & 3 & -3
\end{bmatrix} \sim \begin{bmatrix}
1 & 2 & 0 & 1 \\
0 & 1 & 1 & 6 \\
0 & 0 & 13 & 52
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & 0 & -3 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 4
\end{bmatrix}.
\]

This means \( v = -3p_1 + 2p_2 + 4p_3 \).
Question 3: Is \( W = \{(a, b, c) : a \geq 0\} \) a subspace of \( \mathbb{R}^3 \)? Why?

Clearly \( v = (1, 0, 0) \in W \) but \( -v \) is not a member of \( W \), i.e., the linear combination \( 1 \times 0 + (-1) \times v \) is not in \( W \). Thus \( W \) is not a subspace of \( \mathbb{R}^3 \).

Question 4: Does \( S := \{(1, 1, 1), (1, 0, 1)\} \) form a basis of \( \mathbb{R}^3 \)?

\( \mathbb{R}^3 \) is three-dimensional. The bases of \( \mathbb{R}^3 \) have three vectors. \( S \) contains two vectors only. This cannot be a basis. (Actually it cannot be a spanning set).