Mid term exam 1 (Notes, books, and calculators are not authorized)

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with no justification will not be graded.

**Question 1:** Find the equation of the hyperplane in \( \mathbb{R}^4 \) that passes through \( P = (1, 2, 3, 4) \) and is orthogonal to \( u := (4, 3, 1, 1) \).

The hyperplane in question is the collection of points \( X = (x_1, x_2, x_3, x_4) \) in \( \mathbb{R}^4 \) so that \( X - P \) is orthogonal to \( u \). This means that

\[
0 = (X - P) \cdot u = (x_1 - 1, x_2 - 2, x_3 - 3, x_4 - 4) \cdot (4, 3, 1, 1) = 4x_1 + 3x_2 + x_3 + x_4 - 4 - 6 - 3 - 4
\]

The equation of hyperplane is

\[
4x_1 + 3x_2 + x_3 + x_4 - 20 = 0
\]

**Question 2:** Find an equation of the hyperplane in \( \mathbb{R}^3 \) that passes through \( P = (1, -3, -4) \) and is parallel to the hyperplane \( H' \) determined by the equation \( 3x_1 - 6x_2 + 5x_3 = 2 \).

The definition of \( H' \) implies that \( n = (3, -6, 5) \) is normal to \( H' \). Since \( H \) and \( H' \) are parallel, \( n = (3, -6, 5) \) is also a normal vector to \( H \). By definition \( H \) is the collection of points \( X = (x_1, x_2, x_3) \) in \( \mathbb{R}^3 \) so that \( (X - P) \cdot n = 0 \). This means

\[
0 = (X - P) \cdot (3, -6, 5) = (x_1 - 1, x_2 + 3, x_3 + 4) \cdot (3, -6, 5) =
\]

\[
= 3x_1 - 3 - 6x_2 - 18 + 5x_3 + 20 = 3x_1 - 6x_2 + 5x_3 - 1.
\]

The equation of the hyperplane in question is

\[
3x_1 - 6x_2 + 5x_3 = 1.
\]
Question 3: Let \( A = \begin{bmatrix} 1 + i & -2 - i & 3 + i \\ 4 + 2i & 5 & -6 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 2 \end{bmatrix} \). Find (a) \( A + B \) (b) \( 2A^T - 3B^T \).

\[
\begin{align*}
A + B &= \begin{bmatrix} 2 - i & -2 + i & 2 + i \\ 6 - 2i & 4 & -4 \end{bmatrix} \\
2A^T - 3B^T &= 2 \begin{bmatrix} 1 + i & 4 + 2i \\ -2 - i & 5 \\ 3 - i & -6 \end{bmatrix} - 3 \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 + 2i - 3 & 8 + 4i - 6 \\ -4 - 2i & 10 + 3 \\ 6 - 2i + 3 & -12 - 6 \end{bmatrix} = \begin{bmatrix} -1 + 2i & 2 + 4i \\ -4 - 2i & 13 \\ 9 - 2i & -18 \end{bmatrix}
\end{align*}
\]

Question 4: If possible find the inverses of \( A = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \) and \( B = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \).

The determinant of \( A \) is \(|A| = 10 - 12 = -2 \neq 0 \). The matrix is invertible. We use the formula from class to compute the inverse:

\[
A^{-1} = \frac{1}{-2} \begin{bmatrix} 2 & -4 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} -1 & \frac{2}{3} \\ -\frac{3}{2} & \frac{2}{5} \end{bmatrix}
\]

The determinant of \( B \) is zero. \( B \) is not invertible.
**Question 5:** Consider the matrix \( A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix} \). What is the rank of \( A \) (explain)?

What is the dimension of the null space of \( A \), \( \text{N}(A) \)? (explain).

Clearly

\[
A \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
\]

There are three pivots. This means that the rank of \( A \) is 3. There are two free variable; this means that \( \text{dim}(\text{N}(A)) = 2 \).

**Question 6:** Find \( 2 \times 2 \) nonzero matrices \( A \) and \( B \) such that \( AB \neq BA \).

Consider the following matrices:

\[
A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.
\]

Then

\[
AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = BA.
\]
Question 7: Let \( A = \begin{bmatrix} 2 & 2 & 3 \\ 6 & 10 & 7 \\ 4 & -4 & 15 \end{bmatrix} \). Find the LU factorization of \( A \).

We compute the echelon form of \( A \) to get \( U \) and \( L \):

\[
A = \begin{bmatrix} 2 & 2 & 3 \\ 6 & 10 & 7 \\ 4 & -4 & 15 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & 3 \\ 0 & 4 & -2 \\ 0 & -8 & 9 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & 3 \\ 0 & 4 & -2 \\ 0 & 0 & 5 \end{bmatrix}
\]

\[
l_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \quad l_2 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \quad l_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]

Finally we have the following factorization \( A = LU \) where

\[
L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 2 & 3 \\ 0 & 4 & -2 \\ 0 & 0 & 5 \end{bmatrix}
\]

Question 8: Show that \( S = \{1, 1 - X, (1 - X)^2\} \) is a basis of the vector space \( \mathbb{P}_2(X) \) (over the field \( \mathbb{R} \)).

Let us show that \( \{1, 1 - X, (1 - X)^2\} \) is linearly independent. Assume that there are three real numbers \( x_1, x_2, x_3 \) so that

\[
x_1 + x_2(1 - X) + x_3(1 - X)^2 = 0,
\]

then

\[
x_1 + x_2 + x_3 + (-x_2 - 2x_3)X + x_3X^2 = 0.
\]

The above polynomial is zero iff its coefficients are zero,

\[
x_1 + x_2 + x_3 = 0, \quad -x_2 - 2x_3 = 0 \quad x_3 = 0.
\]

This implies immediately that \( x_1 = x_2 = x_3 = 0 \), thereby proving that \( \{1, 1 - X, (1 - X)^2\} \) is linearly independent. We know that the dimension of \( \mathbb{P}_2(X) \) is 3 (\( \{1, X, X^2\} \) is a basis). As a result \( \{1, 1 - X, (1 - X)^2\} \) is also a basis since this linearly independent set contains 3 elements.
Question 9: Accept as a fact that 

\[ S = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\} \]

is a basis of the vector space of \( 2 \times 2 \) matrices with real coefficients. Find the coordinate vector of \( A = \begin{bmatrix} 2 & 3 \\ 4 & -7 \end{bmatrix} \) relative to \( S \).

We write \( A \) as a linear combination of the basis vectors in \( S \). There are four real numbers \( x_1, x_2, x_3 \) and \( x_4 \) so that

\[
A = x_1 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.
\]

This implies

\[
\begin{bmatrix} 2 & 3 \\ 4 & -7 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + x_3 + x_4 & x_1 - x_2 - x_3 \\ x_1 + x_2 & x_1 \end{bmatrix}
\]

which in turn implies that

\[
x_1 = -7, \quad x_1 + x_2 = 4, \quad x_1 - x_2 - x_3 = 3, \quad x_1 + x_2 + x_3 + x_4 = 2.
\]

We then have \( x_1 = -7, x_2 = 11, x_3 = -21, x_4 = 19 \); as result, the coordinate vector of \( A \) relative to \( S \) is

\[
[A]_S = (-7, 11, -21, 19)
\]

Question 10: Let \( U \) be a \( n \times n \) matrix with coefficients in \( \mathbb{R} \) and assume that the columns of \( U \) are nonzero and orthogonal in \( \mathbb{R}^n \). Prove that \( U \) is invertible.

Let us prove that the null space of \( U \) is zero. Let \( C_1, C_2, \ldots, C_n \) be the columns of \( U \). Let \( X = (x_1, \ldots, x_n) \) be a member of \( N(U) \), i.e., \( NX = 0 \), then

\[
UX = [C_1 \ C_2 \ldots \ C_n]X = x_1C_1 + x_2C_2 + \ldots + x_nC_n = 0.
\]

Taking the dot product of this equation with \( C_1 \) we obtain

\[
x_1C_1 \cdot C_1 + x_2C_2 \cdot C_1 + \ldots + x_nC_n \cdot C_1 = 0.
\]

Since the columns of \( U \) are orthogonal \( C_2 \cdot C_1 = 0, \ldots C_n \cdot C_1 = 0 \), we conclude that

\[
x_1 ||C_1||^2 = 0.
\]

This means \( x_1 = 0 \) since \( C_1 \) is nonzero. Proceed similarly for \( x_2, \ldots, x_n \). This shows that \( X = 0 \), i.e., \( N(U) = \{0\} \), thereby proving that \( U \) is invertible.
Question 11: Let \( C_1 = (2, 1, 4)^T, C_2 = (-3, 2, 1)^T, C_3 = (1, 2, -1)^T \) in \( \mathbb{R}^3 \). (a) Show that \( C_1, C_2, C_3 \) are orthogonal.

We compute \( C_i \cdot C_j \) for \( i \neq j, \, i, j \in \{1, 2, 3\} \).

\[
C_1 \cdot C_2 = -6 + 2 + 4 = 0, \quad C_1 \cdot C_3 = 2 + 2 - 4 = 0, \quad C_2 \cdot C_3 = -3 + 4 - 1 = 0.
\]

This proves the statement.

(b) Consider the \( 3 \times 3 \) matrix with columns \( C_1, C_2, C_3 \), i.e., \( U = [C_1 \, C_2 \, C_3] \). Solve \( UX = V \), where \( V = (10, 6, 4)^T \). (Do not compute the echelon form or the reduced echelon form of the system. Use only the fact that the columns of \( U \) are orthogonal.)

Let \( X = (x_1, x_2, x_3)^T \in \mathbb{R}^3 \) be a solution of \( AX = V \). By definition

\[
V = x_1 C_1 + x_2 C_2 + x_3 C_3
\]

Taking the dot product with \( C_1 \) and using the orthogonality of \( C_1, C_2, C_3 \) gives

\[
V \cdot C_1 = 42 = x_1 C_1 \cdot C_1 + x_2 C_2 \cdot C_1 + x_3 C_3 \cdot C_1 = x_1 \|C_1\|^2 = 21x_1,
\]

i.e., \( x_1 = 2 \). Similarly

\[
V \cdot C_2 = -14 = x_1 C_1 \cdot C_2 + x_2 C_2 \cdot C_2 + x_3 C_3 \cdot C_2 = x_2 \|C_2\|^2 = 14x_2,
\]

i.e., \( x_1 = -1 \).

\[
V \cdot C_3 = 18 = x_1 C_1 \cdot C_3 + x_2 C_2 \cdot C_3 + x_3 C_3 \cdot C_3 = x_3 \|C_3\|^2 = 6x_3,
\]

i.e., \( x_1 = 3 \). Hence the solution is

\[
X = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.
\]