Quizz 4 (Notes, books, and calculators are not authorized)

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded.**

**Question 1:** Are the polynomials \( p_1 = 1 - 3t + 2t^2, \ p_2 = 2 - 4t - t^2, \ p_3 = 1 - 5t + 7t^2 \) linearly independent in \( \mathbb{P}_2(t) \)?

Let \( X = (x_1, x_2, x_3) \in \mathbb{R}^3 \) be so that \( x_1 p_1 + x_2 p_2 + x_3 p_3 = 0 \). Then \( (x_1 + 2x_2 + x_3) + (-3x_1 - 4x_2 - 5x_3)t + (2x_1 - x_2 + 7x_3)t^2 = 0 \). This is equivalent to saying that \( X \) solves

\[
\begin{bmatrix}
1 & 2 & 1 \\
-3 & -4 & -5 \\
2 & -1 & 7
\end{bmatrix} X = 0.
\]

Let us reduce the matrix of the linear system in echelon form

\[
\begin{bmatrix}
1 & 2 & 1 \\
-3 & -4 & -5 \\
2 & -1 & 7
\end{bmatrix} \sim \begin{bmatrix}
1 & 2 & 1 \\
0 & 2 & -2 \\
0 & -5 & 5
\end{bmatrix} \sim \begin{bmatrix}
1 & 2 & 1 \\
0 & 2 & -2 \\
0 & 0 & 0
\end{bmatrix}.
\]

There are only two pivots. There is one free variable. This means that the solution set of the above linear system is not \( \{0\} \). There is some nonzero vector \( X \) so that \( x_1 p_1 + x_2 p_2 + x_3 p_3 = 0 \). This means that the polynomials \( p_1, p_2, p_3 \) are linearly dependent.

**Question 2:** Express the vector \( v = (1, 4, -3) \) as a linear combination of \( v_1 = (1, -2, 5), \ v_2 = (2, -3, 0), \ v_3 = (0, 1, 3) \).

We look for \( (x_1, x_2, x_3) \in \mathbb{R}^3 \) so that \( v = x_1 v_1 + x_2 v_2 + x_3 v_3 \). For all \( t \in \mathbb{R} \) the following holds

\[
(1, 4, -3) = x_1 (1, -2, 5) + x_2 (2, -3, 0) + x_3 (0, 1, 3) = (x_1 + 2x_2, -2x_1 - 3x_2 + x_3, 5x_1 + 3x_3).
\]

This implies

\[
x_1 + 2x_2 = 1, \ 
-2x_1 - 3x_2 + x_3 = 4 \ 
5x_1 + 3x_3 = -3.
\]

We can write this system in the form of an augmented matrix and compute the reduced echelon form:

\[
\begin{bmatrix}
1 & 2 & 0 & 1 \\
-2 & -3 & 1 & 4 \\
5 & 0 & 3 & -3
\end{bmatrix} \sim \begin{bmatrix}
1 & 2 & 0 & 1 \\
0 & 1 & 1 & 6 \\
0 & 0 & 13 & 52
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & 0 & -3 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 4
\end{bmatrix}
\]

This means that \( v = -3v_1 + 2v_2 + 4v_3 \).
**Question 3:** Is \( W = \{at^2 + bt + c : a, b, c \in \mathbb{R}, a + b + c = 0\} \) a subspace of \( \mathbb{P}_2(t) \) over the field \( \mathbb{R} \)? Why?

(i) Clearly \( p = 0 \) is a member of \( W \). (ii) Let \( p_1 = a_1t^2 + b_1t + c_1 \) and \( p_2 = a_2t^2 + b_2t + c_2 \) be two members of \( W \) and let \( \lambda, \mu \in \mathbb{R} \). Then

\[
\lambda p_1 + \mu p_2 = \lambda(a_1t^2 + b_1t + c_1) + \mu(a_2t^2 + b_2t + c_2) = (\lambda a_1 + \mu a_2)t^2 + (\lambda b_1 + \mu b_2)t + (\lambda c_1 + \mu c_2).
\]

Now let us verify if \( \lambda p_1 + \mu p_2 \) is a member of \( W \) by summing the coefficients. The sum of the coefficients of \( \lambda p_1 + \mu p_2 \) is

\[
(\lambda a_1 + \mu a_2) + (\lambda b_1 + \mu b_2) + (\lambda c_1 + \mu c_2) = \lambda(a_1 + b_1 + c_1) + \mu(a_2 + b_2 + c_2) = 0,
\]

thereby proving that \( \lambda p_1 + \mu p_2 \) is a member of \( W \). Items (i) and (ii) prove that \( W \) is a subspace of \( \mathbb{P}_2(t) \).

**Question 4:** Does \( S := \{(1,1,1), (1,0,1), (0,0,1), (0,1,1)\} \) form a basis of \( \mathbb{R}^3 \)?

\( \mathbb{R}^3 \) is three-dimensional. The bases of \( \mathbb{R}^3 \) have three vectors. \( S \) contains four vectors, i.e., one too many. This cannot be a basis. (Actually it cannot be a linearly independent set.)