Quiz 7 (Notes, books, and calculators are not authorized) Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with no justification will not be graded.

**Question 1:** Find the characteristic polynomial of $A = \begin{bmatrix} 2 & 5 \\ 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -2 \\ 9 & -3 \end{bmatrix}$.

\[
P_A(\lambda) := \det(A - \lambda I) = \begin{vmatrix} 2 - \lambda & 5 \\ 4 & 1 - \lambda \end{vmatrix} = (2 - \lambda)(1 - \lambda) - 20 = \lambda^2 - 3\lambda - 18
\]

i.e., $P_A(\lambda) = \lambda^2 - 3\lambda - 18$.

\[
P_B(\lambda) := \det(B - \lambda I) = \begin{vmatrix} 3 - \lambda & -2 \\ 9 & -3 - \lambda \end{vmatrix} = (3 - \lambda)(-3 - \lambda) + 18 = \lambda^2 + 9
\]

i.e., $P_B(\lambda) = \lambda^2 + 9$.

**Question 2:** Consider the real-valued matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & -5 \\ 3 & -5 & 1 \end{bmatrix}$. Is it possible to find a set of three orthogonal real-valued eigenvectors of $A$? Be very accurate and explicit when answering.

Since the matrix is real and symmetric, it is diagonalizable and there exists a set of three real-valued orthogonal vectors. (This is the so-called Spectral Theorem.)
Question 3: Let \( \frac{du}{dt} = G(u) \) be three-dimensional nonlinear dynamical system. Assume that \( u_0 \in \mathbb{R}^3 \) is an equilibrium state, i.e., \( F(u_0) = 0 \) and assume that the dynamics in the neighborhood of \( u_0 \) is described by \( \frac{dz}{dt} = Az \) where \( A \) is \( 3 \times 3 \) real-valued matrix with eigenvalues \( 1 + 3i \), \( 1 - 3i \), and \( -1 \). Is \( u_0 \) a stable point? Be very accurate and explicit when answering.

The real part of the three eigenvalues are \( 1, 1, \) and \( -1 \). The largest real part is \( 1 > 0 \); as a result \( u_0 \) is an unstable point.

Question 4: Is \( A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \) diagonalizable?. Be very accurate and explicit when answering.

Solution 1: The characteristic polynomial is

\[
P_A(\lambda) = (1 - \lambda)^3.
\]

This means that there is only one eigenvalue of multiplicity three. Let \( v = (x, y, z) \) be an eigenvector, then

\[
y + z = 0 \\
z = 0,
\]

which implies that \( y = z = 0 \). The eigenvectors are of the following form \( v = (\alpha, 0, 0), \alpha \in \mathbb{R} \) and \( \alpha \neq 0 \). They all belong to the line \( E_1 := \text{span}\{(1, 0, 0)\} \). As a result, it is not possible to find three independent eigenvectors. The matrix is not diagonalizable.

Solution 2: We deduce as above that \( 1 \) is the only eigenvalue of multiplicity three. If \( A \) was diagonal, we would have \( A = PDP^{-1} \) where \( D = I \). This would mean \( A = PIP^{-1} = PP^{-1} = I \), i.e., \( A = I \), which is obviously wrong. In conclusion \( A \) cannot be diagonalizable.