Quiz 8 (Notes, books, and calculators are not authorized) Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with no justification will not be graded.

**Question 1:** Let $z = x + iy$, were $x,y \in \mathbb{R}$.

(a) Prove that $|e^z| = e^x$.

By definition $e^z = e^x e^{iy}$, where by definition $e^{iy} = \cos(y) + i \sin(y)$. This implies that $|e^z| = |e^x||e^{iy}| = |e^x|$ since $|e^{iy}| = \sqrt{\cos^2(y) + \sin^2(y)} = 1$.

(b) Let $k \in \mathbb{Z}$. Prove that $e^{z+2ik\pi} = e^z$.

By definition $e^{z+2ik\pi} = e^x e^{iy+2ik\pi}$. This implies $e^{z+2ik\pi} = e^x(\cos(y+2k\pi) + i \sin(y+2k\pi)) = e^x(\cos(y) + i \sin(y))$, since the functions $\cos$ and $\sin$ are $2\pi$-periodic.

**Question 2:** Using the notation $z = re^{i\theta}$, were $r \geq 0$, $\theta \in [0,2\pi)$, consider the function $\sqrt{z}$ defined as follows $\sqrt{z} := \sqrt{r}e^{i\frac{\theta}{2}}$. Prove that $\sqrt{z}$ is not continuous across the axis $\{y = 0, x > 0\}$.

Let $z_0 = r$ be an arbitrary number on the positive real axis $\{y = 0, x > 0\}$. Let us set $z_+(\epsilon) = re^{i\epsilon}$, $z_-(\epsilon) = re^{i(2\pi-\epsilon)}$ with $\epsilon > 0$. It is clear that

$$\lim_{\epsilon \to 0} z_+(\epsilon) = z_0 = re^{i2\pi} = \lim_{\epsilon \to 0} z_-(\epsilon),$$

i.e., both $z_+(\epsilon)$ and $z_-(\epsilon)$ converge to $z_0$ as $\epsilon$ goes to zero. If the complex square root function defined above was continuous, the two quantities $\sqrt{z_+}$ and $\sqrt{z_-}$ should converge to the same value as $\epsilon$ goes to zero. But using the above definition of $\sqrt{z}$ we have

$$\lim_{\epsilon \to 0} \sqrt{z_+} = \lim_{\epsilon \to 0} \sqrt{re^{i\frac{\theta}{2}}} = \sqrt{r}$$

and

$$\lim_{\epsilon \to 0} \sqrt{z_-} = \lim_{\epsilon \to 0} \sqrt{re^{i(2\pi-\epsilon)/2}} = \sqrt{re^{i\pi}} = -\sqrt{r}.$$

In conclusion $\lim_{\epsilon \to 0} \sqrt{z_+} \neq \lim_{\epsilon \to 0} \sqrt{z_-}$, thereby proving that $\sqrt{z}$ is not continuous across the axis $\{y = 0, x > 0\}$. 

**Question 3:** Is the function \( z \mapsto f(z) = e^{-x}(x \sin(y) - y \cos(y)) + ie^{-x}(y \sin(y) + x \cos(y)) \), where \( z = x + iy \), holomorphic over \( \mathbb{C} \)? **Justify carefully your answer.**

The real part and imaginary part of \( f \) are clearly differentiable over \( \mathbb{C} \). We just have to verify whether the Cauchy-Riemann conditions hold. Let \( P(x, y) = e^{-x}(x \sin(y) - y \cos(y)) \) and \( Q(x, y) = e^{-x}(y \sin(y) + x \cos(y)) \)

\[
\begin{align*}
\partial_x P(x, y) &= e^{-x}(-x \sin(y) + y \cos(y) + \sin(y)) \\
\partial_y Q(x, y) &= e^{-x}(\sin(y) + y \cos(y) - x \sin(y))
\end{align*}
\]

and

\[
\begin{align*}
\partial_y P(x, y) &= e^{-x}(x \cos(y) - \cos(y) + y \sin(y)) \\
\partial_x Q(x, y) &= e^{-x}(-y \sin(y) - x \cos(y) + \cos(y))
\end{align*}
\]

The two relations \( \partial_x P(x, y) = \partial_y Q(x, y) \) and \( \partial_y P(x, y) = -\partial_x Q(x, y) \) are satisfied.

**Question 4:** (a) Is the function \( z \mapsto f(z) = x - 2iy \), where \( z = x + iy \), holomorphic? **Justify carefully your answer.**

Verify whether the Cauchy-Riemann conditions hold. Let \( x = P(x, y) = \Re(f(z)) \) and \( -2y = Q(x, y) = \Im(f(z)) \). Then

\[
\partial_x P(x, y) = 1 \neq -2 = \partial_y Q(x, y).
\]