Quiz 3 (Notes, books, and calculators are not authorized)

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with no justification will not be graded.

**Question 1**: Consider the system \[
\begin{bmatrix}
a & 9 \\
1 & a
\end{bmatrix}
\begin{bmatrix}
X
\end{bmatrix}
= \begin{bmatrix}
3 \\
b
\end{bmatrix}.
\]

(a) For which values of \(a\) does the system have a unique solution?

The system has a unique solution if and only if the determinant is non zero:

\[
\begin{vmatrix}
a & 9 \\
1 & a
\end{vmatrix} = a^2 - 9 \neq 0
\]

This is equivalent to \(a \neq \pm 3\). The admissible values are \(a \in \mathbb{R}\setminus\{-3,3\}\).

(b) Find those values of \((a, b)\) for which the system has no solution.

Observe first that it is necessary that \(a = 3\) or \(a = -3\) for the above system not to have a solution (otherwise the determinant is nonzero). In particular, \(a\) cannot be zero. Then, the augmented matrix is reduced in echelon form as follows:

\[
M \sim \begin{bmatrix}
a & 9 & 3 \\
1 & a & b
\end{bmatrix} \sim \begin{bmatrix}
a & 9 & 3 \\
0 & a^2 - 9 & ab - 3
\end{bmatrix}.
\]

The system has no solution iff \(a^2 - 9 = 0\) and \(ab - 3 \neq 0\). This means \(a = \pm 3\) and \(b \neq \pm 1\). In conclusion the above system has no solution if and only if either \(a = 3\) and \(b \neq 1\), or \(a = -3\) and \(b \neq -1\).

\[-6x_3 + 2x_1 + 2x_2 = 1\]

**Question 2**: Consider the following linear system

\[
\begin{align*}
-6x_3 + 2x_1 + 2x_2 &= 1 \\
3x_2 + 4x_1 &= 8 \\
2x_2 &= 4
\end{align*}
\]

(a) Write the augmented matrix.

The system can be rewritten into the following form:

\[
\begin{align*}
2x_1 + 2x_2 - 6x_3 &= 1 \\
4x_1 + 3x_2 + 0x_3 &= 8 \\
0x_1 + 2x_2 + 0x_3 &= 4
\end{align*}
\]

and the augmented matrix is

\[
\begin{bmatrix}
2 & 2 & -6 & 1 \\
4 & 3 & 0 & 8 \\
0 & 2 & 0 & 4
\end{bmatrix}
\]

(b) Determine the pivot and free variables.

We reduce the augmented matrix in echelon form.

\[
\begin{bmatrix}
2 & 2 & -6 & 1 \\
4 & 3 & 0 & 8 \\
0 & 2 & 0 & 4
\end{bmatrix} \sim \begin{bmatrix}
2 & 2 & -6 & 1 \\
0 & -1 & 12 & 6 \\
0 & 2 & 0 & 4
\end{bmatrix} \sim \begin{bmatrix}
0 & -1 & 12 & 6 \\
0 & 0 & 24 & 16 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

There is no free variable. There are three pivots, which are 2, -1 and 24 in the first, second and third column, respectively.
Question 3: (a) Compute the echelon form of the augmented matrix \( M = \begin{bmatrix} 2 & 2 & 1 & 1 & 3 \\ 2 & 2 & 3 & 5 & 9 \\ 6 & 6 & 4 & 6 & 13 \end{bmatrix} \)

\[
M = \begin{bmatrix} 2 & 2 & 1 & 1 & 3 \\ 2 & 2 & 3 & 5 & 9 \\ 6 & 6 & 4 & 6 & 13 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & 1 & 1 & 3 \\ 0 & 0 & 2 & 4 & 6 \\ 0 & 0 & 1 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & 1 & 1 & 3 \\ 0 & 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}
\]

(b) Compute the reduced echelon form of \( M \).

\[
M \sim \begin{bmatrix} 2 & 2 & 1 & 1 & 3 \\ 0 & 0 & 2 & 4 & 6 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & 0 & -1 & 0 \\ 0 & 0 & 2 & 4 & 6 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & 0 & 0 & 1 \\ 0 & 0 & 2 & 4 & 6 \\ 0 & 0 & 1 & 1 \end{bmatrix}
\]

The reduced echelon form of \( M \) is

\[
M \sim \begin{bmatrix} 1 & 1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}
\]

Question 4: Let \( A \) be a \( m \times n \) matrix and \( B \) a \( n \times p \) matrix. (a) Show that if \( B \) has a zero column then \( AB \) has a zero column.

Let \( C_1, \ldots, C_p \) be the columns of \( B \), i.e., \( B = [C_1 \ C_2 \ldots C_p] \). The definition of the product \( AB \) implies that

\[
AB = [AC_1 \ AC_2 \ldots AC_p],
\]

in other words, the columns of the matrix \( AB \) are \( AC_1, AC_2, \ldots AC_p \). If column \( C_j \) is zero then \( AC_j \) is also zero, meaning that the \( j \)th column of \( AB \) is zero.

(b) Show that if \( A \) has a zero row then \( AB \) has a zero row. (Work with \( B^T A^T \) and use (a).)

Let \( R_1, \ldots R_m \) be the rows of \( A \). Then the columns of \( A^T \) are \( R_1^T, \ldots, R_m^T \). If \( R_j^T \) is zero then from (a) we know that the \( j \)th column of \( B^T A^T \) is also zero. This also means that the \( j \)th row of \( (B^T A^T)^T = AB \) is zero.