Quiz 5 (Notes, books, and calculators are not authorized)

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with no justification will not be graded. \( P_2 \) is the vector space over \( \mathbb{R} \) of polynomials of degree at most two.

**Question 1:** Consider the following two bases of \( P_2 \): \( E = \{(1 - t), t, 4t(1 - t)\} \) and \( F = \{(1 - 2t)(1 - t), t(2t - 1), 4t(1 - t)\} \). Compute the change of basis matrix \( P_{F \rightarrow E} \).

Let \( e_1 = 1 - t \), \( e_2 = t \), \( e_3 = 4t(1 - t) \), \( f_1 = (1 - 2t)(1 - t) \), \( f_2 = t(2t - 1) \), \( f_3 = 4t(1 - t) \). The definition of the matrix \( P_{F \rightarrow E} \) is

\[
P_{F \rightarrow E} = \begin{bmatrix} f_1 \mid E & f_2 \mid E & f_3 \mid E \end{bmatrix}.
\]

We have

\[
f_1 = (1 - 2t)(1 - t) = (1 - t) - 2t(1 - t) = e_1 - \frac{1}{2} e_3
\]

\[
f_2 = t(2t - 1) = t + 2t(t - 1) = e_2 - \frac{1}{2} e_3
\]

\[
f_3 = e_3.
\]

As a result

\[
P_{F \rightarrow E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}.
\]

**Question 2:** Consider the following basis of \( P_2 \): \( C = \{1, t, t^2\} \). Let \( F : P_2 \rightarrow P_2 \) be the linear mapping such that \( F(p) = p(t) + \frac{1}{6} t^2 p'(t) - \int_0^t p(\tau)d\tau \) for all \( p \in P_2 \). Compute \( [F]_{C,C} \).

Let us denote \( e_1 = 1 \), \( e_2 = t \), \( e_3 = t^2 \). The definition of \( [F]_{C,C} \) is

\[
[F]_{C,C} = \begin{bmatrix} [F(e_1)]_C & [F(e_2)]_C & [F(e_3)]_C \end{bmatrix}.
\]

We have

\[
F(e_1) = c_1(t) + \frac{1}{6} t^2 c'_1(t) - \int_0^t c_1(\tau)d\tau = 1 - \int_0^t d\tau = 1 - t.
\]

\[
F(e_2) = c_2(t) + \frac{1}{6} t^2 c'_2(t) - \int_0^t c_2(\tau)d\tau = t + \frac{1}{6} t^2 - \int_0^t \tau d\tau = t - \frac{1}{3} t^2.
\]

\[
F(e_3) = c_3(t) + \frac{1}{6} t^2 c'_3(t) - \int_0^t c_3(\tau)d\tau = t^2 + \frac{1}{3} t^3 - \int_0^t \tau^2 d\tau = t^2.
\]

As a result

\[
[F]_{C,C} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -\frac{1}{3} & 1 \end{bmatrix}.
\]
Question 3: Consider the following basis of $\mathbb{P}_2$: $C = \{1, t, t^2\}$. Let $G : \mathbb{P}_2 \rightarrow \mathbb{P}_2$ be the linear mapping such that $[G]_{C,C} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & 1 \end{bmatrix}$. Let $p = a + bt + ct^2$ be an arbitrary vector in $\mathbb{P}_2$. Compute $G(p)$.

The definition of $[G]_{C,C}$ is such that

$$ [G(p)]_C = [G]_{C,C}[p]_C. $$

As a result

$$ [G(p)]_C = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ a+b \\ \frac{1}{3}a + \frac{1}{3}b + c \end{bmatrix}. $$

This means that

$$ G(p) = a + (a+b)t + \left(\frac{1}{3}a + \frac{1}{3}b + c\right)t^2. $$

Question 4: Let $H : \mathbb{P}_2 \rightarrow \mathbb{P}_2$ be the linear mapping such that $H(a + bt + ct^2) = a + (a+b)t + \left(\frac{1}{3}a + \frac{1}{3}b + c\right)t^2$. Let $F$ be the mapping defined in Question 2. Compute $F(H(p))$, for all $p = a + bt + ct^2$.

(Long answer): The definitions of $F$ and $H$ imply that

$$ F(H(p)) = H(p) + \frac{1}{6}t^2(H(p))' - \int_0^t H(\tau)d\tau $$

$$ = a + (a+b)t + \left(\frac{1}{3}a + \frac{1}{3}b + c\right)t^2 + \frac{1}{6}t^2(\frac{1}{3}a + \frac{1}{3}b + c)t $$

$$ - at - \frac{1}{2}(a+b)t^2 - \frac{1}{3}(\frac{1}{3}a + \frac{1}{3}b + c)t^3 $$

$$ = a + bt + ct^2 = p $$

In conclusion $F(H(p)) = p$, i.e., $H$ is the inverse of $F$.

(Short answer): Observe that $G$ and $H$ are the same operators. Observe also that $[F]_C[H]_C = [F]_C[G]_C = I$, as a result $F(G(p)) = p$. 
