Question 1: Consider the wave equation
\[ \partial_{tt}w - 4\partial_{xx}w = 0, \quad x \in (-\infty, +\infty), \quad t > 0, \]
with initial data \( w(x, 0) = \frac{1}{1 + x^2}, \quad \partial_t w(x, 0) = -\frac{4x}{(1 + x^2)^2}. \) Compute the solution \( w(x, t). \)

The wave speed is 2. The solution is given by the D’Alembert formula,

\[
w(x, t) = \frac{1}{2} \left( \frac{1}{1 + (x - 2t)^2} + \frac{1}{1 + (x + 2t)^2} \right) + \frac{1}{4} \int_{x-2t}^{x+2t} \frac{4\tau}{(1 + \tau^2)^2} d\tau
\]

After integration, we obtain

\[
= \frac{1}{2} \left( \frac{1}{1 + (x - 2t)^2} + \frac{1}{1 + (x + 2t)^2} \right) + \frac{1}{4} \left[ \frac{2}{(1 + \tau^2)} \right]_{x-2t}^{x+2t},
\]

which finally gives

\[
w(x, t) = \frac{1}{1 + (x + 2t)^2}.
\]
Question 2: Use the Fourier transform method to solve the equation $\partial_t u + \frac{2t}{1+t^2} \partial_x u = 0$, $u(x,0) = u_0(x)$, in the domain $x \in (-\infty, +\infty)$ and $t > 0$.

We take the Fourier transform of the equation with respect to $x$:

$$
0 = \partial_t \mathcal{F}(u) + \mathcal{F}\left( \frac{2t}{1+t^2} \partial_x u \right)
= \partial_t \mathcal{F}(u) + \frac{2t}{1+t^2} \mathcal{F}(\partial_x u)
= \partial_t \mathcal{F}(u) - i\omega \frac{2t}{1+t^2} \mathcal{F}(u).
$$

This is a first-order linear ODE:

$$
\frac{\partial_t \mathcal{F}(u)}{\mathcal{F}(u)} = i\omega \frac{2t}{1+t^2} = i\omega \frac{d}{dt}(\log(1 + t^2))
$$

The solution is

$$
\mathcal{F}(u)(\omega, t) = K(\omega)e^{i\omega \log(1+t^2)}.
$$

Using the initial condition, we obtain

$$
\mathcal{F}(u_0)(\omega) = \mathcal{F}(u)(\omega, 0) = K(\omega).
$$

The shift lemma (see formula (??)) implies

$$
\mathcal{F}(u)(\omega, t) = \mathcal{F}(u_0)(\omega)e^{i\omega \log(1+t^2)} = \mathcal{F}(u_0(x - \log(1 + t^2))),
$$

Applying the inverse Fourier transform finally gives

$$
u(x, t) = u_0(x - \log(1 + t^2)).$$
Question 3: Consider the wave equation \( \partial_{tt}w - \partial_{xx}w = 0, \ x \in (0, 4), \ t > 0, \) with

\[
w(x, 0) = f(x), \quad x \in (0, 4), \quad \partial_t w(x, 0) = 0, \quad x \in (0, 4), \quad \text{and} \quad \partial_x w(0, t) = 0, \quad \partial_x w(4, t) = 0, \quad t > 0.
\]

where \( f(x) = x - 1, \) if \( x \in [1, 2], \) \( f(x) = 3 - x, \) if \( x \in [2, 3], \) and \( f(x) = 0 \) otherwise. Give a simple expression of the solution in terms of an extension of \( f. \) Give a graphical solution to the problem at \( t = 0, \ t = 1, \ t = 2, \) and \( t = 3 \) (draw four different graphs and explain).

We know from class that with homogeneous Neumann boundary conditions, the solution to this problem is given by the D'Alembert formula where \( f \) must be replaced by the periodic extension (of period 8) of its even extension, say \( f_{e,p}, \) where

\[
f_{e,p}(x + 8) = f_{e,p}(x), \quad \forall x \in \mathbb{R}
\]

\[
f_{e,p}(x) = \begin{cases} f(x) & \text{if } x \in [0, 4] \\ f(-x) & \text{if } x \in [-4, 0] \end{cases}
\]

The solution is

\[
u(x, t) = \frac{1}{2}(f_{e,p}(x - t) + f_{e,p}(x + t)).
\]

I draw on the left of the figure the graph of \( f_{e,p}. \) Half the graph moves to the right with speed 1, the other half moves to the left with speed 1.

(a) Initial data + periodic extension of the even extension at \( t = 0, 1, 2, 3. \) Solid line waves move to the right, dotted line waves move to the left

(b) Solution in domain \( (0, 4) \) at \( t = 0, 1, 2, 3 \)
Question 4: Solve the integral equation:
\[ \int_{-\infty}^{+\infty} \left( f(y) - \sqrt{2} e^{-\frac{y^2}{2\pi}} - \frac{1}{1 + y^2} \right) f(x-y) \, dy = - \int_{-\infty}^{+\infty} \frac{\sqrt{2}}{1 + y^2} e^{-\frac{(x-y)^2}{2\pi}} \, dy, \quad \forall x \in (-\infty, +\infty). \]

(Hint: there is an easy factorization after applying the Fourier transform.)

The equation can be re-written

\[ f \ast (f - \sqrt{2} e^{-\frac{y^2}{2\pi}} - \frac{1}{1 + x^2}) = - \frac{1}{1 + x^2} \ast \sqrt{2} e^{-\frac{x^2}{2\pi}}. \]

We take the Fourier transform of the equation and apply the Convolution Theorem (see (??))

\[ 2\pi \mathcal{F}(f) \left( \mathcal{F}(f) - \sqrt{2} \mathcal{F}(e^{-\frac{x^2}{2\pi}}) - \mathcal{F}(\frac{1}{1 + x^2}) \right) = -2\pi \mathcal{F}(\frac{1}{1 + x^2}) \sqrt{2} \mathcal{F}(e^{-\frac{x^2}{2\pi}}) \]

Solution 1: Using (??), (??) we obtain

\[ \sqrt{2} \mathcal{F}(e^{-\frac{x^2}{2\pi}}) = \sqrt{2} \frac{1}{\sqrt{4\pi^2}} e^{-\frac{\omega^2}{4\pi^2}} = e^{-\frac{\omega^2}{2\pi^2}} \]

\[ \mathcal{F}(\frac{1}{1 + x^2}) = \frac{1}{2} e^{-|\omega|}, \]

which gives

\[ \mathcal{F}(f) \left( \mathcal{F}(f) - e^{-\frac{\omega^2}{2\pi^2}} - \frac{1}{2} e^{-|\omega|} \right) = - \frac{1}{2} e^{-|\omega|} e^{-\frac{\omega^2}{2\pi^2}}. \]

This equation can also be re-written as follows

\[ \mathcal{F}(f)^2 - \mathcal{F}(f)e^{-\frac{\omega^2}{2\pi^2}} - \mathcal{F}(f) \frac{1}{2} e^{-|\omega|} + \frac{1}{2} e^{-|\omega|} e^{-\frac{\omega^2}{2\pi^2}} = 0, \]

and can be factorized as follows:

\[ (\mathcal{F}(f) - e^{-\frac{\omega^2}{2\pi^2}})(\mathcal{F}(f) - \frac{1}{2} e^{-|\omega|}) = 0. \]

This means that either \( \mathcal{F}(f) = e^{-\frac{\omega^2}{2\pi^2}} \) or \( \mathcal{F}(f) = \frac{1}{2} e^{-|\omega|} \). Taking the inverse Fourier transform, we finally obtain two solutions

\[ f(x) = \sqrt{2} e^{-\frac{x^2}{2\pi}}, \quad \text{or} \quad f(x) = \frac{1}{1 + x^2}. \]

Solution 2: Another solution consists of remarking that the equation with the Fourier transform can be rewritten as follows:

\[ \mathcal{F}(f)^2 - \mathcal{F}(\sqrt{2} e^{-\frac{x^2}{2\pi}}) \mathcal{F}(f) - \mathcal{F}(\frac{1}{1 + x^2}) \mathcal{F}(f) + \mathcal{F}(\sqrt{2} e^{-\frac{x^2}{2\pi}}) \mathcal{F}(\frac{1}{1 + x^2}) = 0, \]

which can factorized as follows:

\[ (\mathcal{F}(f) - \mathcal{F}(\sqrt{2} e^{-\frac{x^2}{2\pi}})) (\mathcal{F}(f) - \mathcal{F}(\frac{1}{1 + x^2})) = 0. \]

The either \( \mathcal{F}(f) = \mathcal{F}(\sqrt{2} e^{-\frac{x^2}{2\pi}}) \) or \( \mathcal{F}(f) = \mathcal{F}(\frac{1}{1 + x^2}) \). The conclusion follows easily.
Question 5: Consider the equation \( \partial_{xx}u(x) = f(x) \), \( x \in (0, L) \), with \( u(0) = a \) and \( \partial_x u(L) = b \).

(a) Compute the Green’s function of the problem.

Let \( x_0 \) be a point in \((0, L)\). The Green’s function of the problem is such that

\[ \partial_{xx} G(x, x_0) = \delta_{x_0}, \quad G(0, x_0) = 0, \quad \partial_x G(L, x_0) = 0. \]

The following holds for all \( x \in (0, x_0) \):

\[ \partial_{xx} G(x, x_0) = 0. \]

This implies that \( G(x, x_0) = ax + b \) in \((0, x_0)\). The boundary condition \( G(0, x_0) = 0 \) gives \( b = 0 \).

Likewise, the following holds for all \( x \in (x_0, L) \):

\[ \partial_{xx} G(x, x_0) = 0. \]

This implies that \( G(x, x_0) = cx + d \) in \((x_0, L)\). The boundary condition \( \partial_x G(L, x_0) = 0 \) gives \( c = 0 \). The continuity of \( G(x, x_0) \) at \( x_0 \) implies that \( ax_0 = d \). The condition

\[ \int_{-\epsilon}^{\epsilon} \partial_{xx} G(x, x_0) dx = 1, \quad \forall \epsilon > 0, \]

gives the so-called jump condition: \( \partial_x G(x_0^+, x_0) - \partial_x G(x_0^-, x_0) = 1 \). This means that \( 0 - a = 1 \), i.e., \( a = -1 \) and \( d = -x_0 \). In conclusion

\[ G(x, x_0) = \begin{cases} -x & \text{if } x \leq x_0, \\ -x_0 & \text{otherwise}. \end{cases} \]

(b) Give the integral representation of \( u \) using the Green’s function.

Let \( x_0 \) be a point in \((0, L)\). The definition of the Dirac measure at \( x_0 \) is such that

\[ u(x_0) = \langle \delta_{x_0}, u \rangle = \langle \partial_{xx} G(\cdot, x_0), u \rangle \]

\[ = -\int_0^L \partial_x G(x, x_0) \partial_{xx} u(x) dx + [\partial_x G(x, x_0) u(x)]_0^L \]

\[ = \int_0^L G(x, x_0) \partial_{xx} u(x) dx - [G(x, x_0) \partial_{xx} u(x)]_0^L + [\partial_x G(x, x_0) u(x)]_0^L \]

\[ = \int_0^L G(x, x_0) f(x) dx - G(L, x_0) \partial_x u(L) + G(0, x_0) \partial_x u(0) + \partial_x G(L, x_0) u(L) - \partial_x G(0, x_0) u(0). \]

This finally gives the following representation of the solution:

\[ u(x_0) = \int_0^L G(x, x_0) f(x) dx - G(L, x_0) b - \partial_x G(0, x_0) a \]
**Question 6:** Consider the wave equation with variable coefficients

\[ m(x) \partial_{tt} u(x, t) - \partial_x \left( \mu(x) \partial_x u(x, t) \right) = 0, \quad x \in (0, L), \quad t > 0, \]

with boundary condition \( u(0, t) = 0, \partial_x u(L, t) = 0, \) where \( m \) (density of the material) and \( \mu, \) (elasticity of the material) and smooth positive functions. Assuming that the solution \( u(x, t) \) is smooth, prove that the quantity

\[ \int_0^L \left( \frac{1}{2} m(x) |\partial_t u(x, t)|^2 + \frac{1}{2} \mu(x) |\partial_x u(x, t)|^2 \right) \, dx \]

is independent of \( t. \) (Hint: energy argument with \( \partial_t u(x, t) \) and use \( a(t) \partial_t a(t) = \partial_t \left( \frac{1}{2} a(t)^2 \right). \))

Multiply the equation by \( \partial_t (ux, t) \) and integrate over \( (0, L). \)

\[ 0 = \int_0^L \left( m(x) \partial_{tt} u(x, t) \partial_t u(x, t) - \partial_x \mu(x) \partial_x u(x, t) \partial_t u(x, t) \right) \, dx \]

Integrate by parts, use the boundary conditions, and use twice the formula \( a(t) \partial_t a(t) = \partial_t \left( \frac{1}{2} a(t)^2 \right), \)

\[ 0 = \int_0^L \left( m(x) \partial_t \left( \frac{1}{2} \partial_t u(x, t) \right)^2 + \mu(x) \partial_x u(x, t) \partial_t \partial_x u(x, t) \right) \, dx \]

\[ = \int_0^L \left( m(x) \partial_t \left( \frac{1}{2} \partial_t u(x, t) \right)^2 + \mu(x) \partial_x \left( \frac{1}{2} \partial_x u(x, t) \right)^2 \right) \, dx. \]

Using the fact that \( m \) and \( \mu \) are time-independent, we also have

\[ 0 = \int_0^L \left( \partial_t \left( \frac{1}{2} m(x) |\partial_t u(x, t)|^2 \right) + \partial_t \left( \frac{1}{2} \mu(x) |\partial_x u(x, t)|^2 \right) \right) \, dx, \]

\[ = \frac{d}{dt} \int_0^L \left( \frac{1}{2} m(x) |\partial_t u(x, t)|^2 + \frac{1}{2} \mu(x) |\partial_x u(x, t)|^2 \right) \, dx, \]

which proves the statement.