In the previous chapter, we presented conditions to assert the well-posedness of the following model problem:

\[
\begin{align*}
\text{Seek } u & \in V \text{ such that } \\
a(u, w) & = \ell(w), \quad \forall w \in W,
\end{align*}
\] (16.1)

where \( V \) and \( W \) are Banach spaces, \( a \) is a bounded bilinear (or sesquilinear) form on \( V \times W \), and \( \ell \) is a bounded (anti)linear form on \( W \). In this chapter, we study the approximation of (16.1) by the Galerkin method. The starting point is to replace the infinite-dimensional spaces \( V \) and \( W \) in (16.1) by finite-dimensional spaces \( V_h \) and \( W_h \), and possibly the forms \( a \) and \( \ell \) by some approximate forms \( a_h \) and \( \ell_h \) defined on \( V_h \times W_h \) and \( W_h \), respectively. The spaces \( V_h \) and \( W_h \) are finite element spaces constructed using the techniques presented in the previous chapters. In this chapter, we derive the main results for asserting the well-posedness of the approximate problem, leading to discrete versions of the Lax–Milgram Lemma and the BNB Theorem. We also present a useful tool, known as Fortin’s criterion, to infer a discrete inf-sup condition from a similar condition at the continuous level. Finally, we derive a bound on the approximation error \( (u - u_h) \) in a simple setting; see Chapter 17 for a thorough error analysis.

### 16.1 Setting

The discrete problem takes the following form:

\[
\begin{align*}
\text{Seek } u_h & \in V_h \text{ such that } \\
a_h(u_h, w_h) & = \ell_h(w_h), \quad \forall w_h \in W_h,
\end{align*}
\] (16.2)

where \( a_h \) is a bounded bilinear (or sesquilinear) form on \( V_h \times W_h \) and \( \ell_h \) is a bounded (anti)linear form on \( W_h \). Since the spaces \( V_h \) and \( W_h \) are finite-dimensional, (16.2) is called a discrete problem.
Definition 16.1 (Trial and test space). The space \( V_h \) is called the solution space (or the trial space), and the space \( W_h \) the test space.

In general, these spaces need not be subspaces of \( V \) and \( W \), respectively. We introduce the following terminology for the particular situation (often considered in what follows) where subspaces are considered.

Definition 16.2 (Conforming approximation). The approximation is said to be conforming if \( V_h \subset V \) and \( W_h \subset W \). The terminology internal approximation is also found in the literature.

Another useful distinction is whether the same discrete space is used as solution and test space or not.

Definition 16.3 (Standard Galerkin, Petrov–Galerkin). The Galerkin approximation is said to be standard when \( W_h = V_h \) in the discrete problem (16.2). When the two discrete spaces are different, (16.2) is called a Petrov–Galerkin approximation or a nonstandard Galerkin approximation.

16.2 Discrete well-posedness

Our goal is to identify conditions to assert the well-posedness of the discrete problem (16.2). We equip the discrete spaces \( V_h \) and \( W_h \) with norms denoted \( \| \cdot \|_{V_h} \) and \( \| \cdot \|_{W_h} \). In finite dimension, all the norms are equivalent, and (bi)linear forms are always bounded. However, the choice of the norms with which to equip the discrete spaces \( V_h \) and \( W_h \) is important in view of error analysis. Indeed, what matters is the behavior of the derived error bounds as the discretization parameter \( h \) tends to zero. It is therefore essential to avoid norm equivalence constants that depend on discretization parameters.

16.2.1 Discrete Lax–Milgram

Lemma 16.4 (Discrete Lax–Milgram). Let \( V_h \) be a finite-dimensional space over \( \mathbb{R} \). Set \( W_h = V_h \) in problem (16.2). Let \( a_h \) be a bounded bilinear form on \( V_h \times V_h \) and let \( \ell_h \) be a bounded linear form on \( V_h \). Assume that the bilinear form \( a_h \) is coercive on \( V_h \), i.e., there is a real number \( \alpha_h > 0 \) and \( \xi = \pm 1 \) such that

\[
\xi a_h(v_h, v_h) \geq \alpha_h \| v_h \|_{V_h}^2, \quad \forall v_h \in V_h. \tag{16.3}
\]

Then, problem (16.2) is well-posed and the a priori estimate \( \| u_h \|_{V_h} \leq (\alpha_h)^{-1} \| \ell_h \|_{V_h^*} \) holds. In the complex case, the same conclusions hold if there is a real number \( \alpha_h > 0 \) and a complex number \( \xi \) with \( |\xi| = 1 \) such that

\[
\Re (\xi a_h(v_h, v_h)) \geq \alpha_h \| v_h \|_{V_h}^2, \quad \forall v_h \in V_h. \tag{16.4}
\]
Proof. We give the proof in the real case; the complex case is treated similarly. Let \( A_h : V_h \to V_h' \) be the linear operator such that \( \langle A_h v_h, w_h \rangle_{V_h'} = a_h(v_h, w_h) \) for all \( v_h, w_h \in V_h \). Well-posedness of (16.2) is equivalent to \( A_h \) being injective, i.e., \( \ker(A_h) = \{0\} \), since \( \dim(V_h) = \dim(V_h') < \infty \). Let \( v_h \in \ker(A_h) \), then \( 0 = \xi \langle A_h v_h, v_h \rangle_{V_h'} = \xi a_h(v_h, v_h) \geq \alpha_h \|v_h\|^2_{V_h} \), which proves that \( v_h = 0 \). Hence, \( \ker(A_h) = \{0\} \).

Remark 16.5 (Variational formulation). As in the continuous setting, see Proposition 15.8, if \( a_h \) is coercive (with \( \xi = 1 \)) and symmetric in the real case, \( u_h \) solves (16.2) with \( W_h = V_h \) if and only if \( u_h \) minimizes the functional

\[
\mathcal{E}_h(v_h) = \frac{1}{2} a_h(v_h, v_h) - \ell_h(v_h)
\]

over \( V_h \). In the conforming case where \( V_h \subset V \), \( a_h = a \), and \( \ell_h = \ell \), we observe that \( \mathcal{E}_h(u_h) = \mathcal{E}(u_h) \geq \mathcal{E}(u) \).

\[\square\]

16.2.2 Discrete BNB

Theorem 16.6 (Discrete BNB). Let \( V_h \) and \( W_h \) be finite-dimensional spaces over \( \mathbb{R} \). Let \( a_h \) be a bilinear form on \( V_h \times W_h \) and let \( \ell_h \) be a linear form on \( W_h \). Then, problem (16.2) is well-posed if and only if

\[
\inf_{v_h \in V_h, w_h \in W_h} \sup_{v_h \in V_h, w_h \in W_h} \frac{a_h(v_h, w_h)}{\|v_h\|_{V_h} \|w_h\|_{W_h}} =: \alpha_h > 0, \quad (16.5a)
\]

\[
\dim(V_h) = \dim(W_h). \quad (16.5b)
\]

Moreover, the a priori estimate \( \|u_h\|_{V_h} \leq (\alpha_h)^{-1} \|\ell_h\|_{W_h'} \) holds. Condition (16.5a) is equivalent to the following inf-sup condition:

\[
\exists \alpha_h > 0, \quad \alpha_h \|v_h\|_{V_h} \leq \sup_{w_h \in W_h} \frac{a_h(v_h, w_h)}{\|w_h\|_{W_h}}, \quad \forall v_h \in V_h. \quad (16.6)
\]

In the complex case, the real part of the numerator is taken in (16.5a).

Proof. We give the proof in the real case; the complex case is treated similarly. Let \( A_h : V_h \to W_h' \) be the linear operator such that

\[
\langle A_h v_h, w_h \rangle_{W_h'} = a_h(v_h, w_h), \quad \forall (v_h, w_h) \in V_h \times W_h. \quad (16.7)
\]

The well-posedness of (16.2) is equivalent to \( A_h \) being an isomorphism, which owing to the finite-dimensional setting and the Rank Theorem, is equivalent to the following conditions:

\[
\ker(A_h) = \{0\} \text{ (i.e., } A_h \text{ is injective)}, \quad (16.8a)
\]

\[
\dim(V_h) = \dim(W_h'). \quad (16.8b)
\]

Since \( \dim(W_h) = \dim(W_h') \), (16.5b) is equivalent to (16.8b). Let us prove that (16.8a) is equivalent to the inf-sup condition (16.5a). Assume first that (16.5a) holds and let \( v_h \in V_h \) be such that \( A_h v_h = 0 \). Then,
\[
\alpha_h \| v_h \|_{V_h} \leq \sup_{w_h \in W_h} \frac{\langle A_h v_h, w_h \rangle_{W'_h, W_h}}{\| w_h \|_{W_h}} = 0,
\]
whence \( v_h = 0 \). Conversely, assume \( \ker(A_h) = \{0\} \) and let us prove the inf-sup condition \((16.5a)\). Reasoning by contradiction, consider a sequence \( (v_{hn})_{n \in \mathbb{N}} \) in \( V_h \) with \( \| v_{hn} \|_{V_h} = 1 \) and
\[
\sup_{w_h \in W_h} \frac{\langle A_h v_{hn}, w_h \rangle_{W'_h, W_h}}{\| w_h \|_{W_h}} \leq \frac{1}{n}.
\]
Since \( V_h \) is finite-dimensional, its unit sphere is compact. Therefore, up to a subsequence, \( v_{hn} \) converges to a certain \( v_h \) in \( V_h \). The limit \( v_h \) satisfies \( \| v_h \|_{V_h} = 1 \) and \( a_h(v_h, w_h) = 0 \) for all \( w_h \in W_h \). The latter property implies \( v_h \in \ker(A_h) \), but \( \ker(A_h) = \{0\} \) which contradicts \( \| v_h \|_{V_h} = 1 \). Finally, the a priori estimate is proved as in the continuous case. \( \square \)

**Remark 16.7 (Verifying \((16.5)\)).** Since \((16.5b)\) is, in general, simple to verify, the key property for discrete well-posedness is \((16.5a)\). \( \square \)

**Remark 16.8 (Discrete vs. continuous BNB Theorems).** The inf-sup condition \((16.5a)\) is nothing but the first condition (BNB1) of the BNB Theorem 15.10. The second condition (BNB2) amounts to
\[
\forall w_h \in W_h, \quad [a_h(v_h, w_h) = 0, \forall v_h \in V_h] \implies [w_h = 0]. \quad (16.9)
\]
Let us introduce the adjoint operator \( A_h^* : W_h \to V'_h \) (note that \( W_h \) is reflexive, being finite-dimensional) such that
\[
\langle A_h^* w_h, v_h \rangle_{V'_h, V_h} = a_h(v_h, w_h), \quad \forall (v_h, w_h) \in V_h \times W_h. \quad (16.10)
\]
Then, \((16.9)\) amounts to \( A_h^* \) being injective, and this statement is equivalent to \((16.5b)\) provided \((16.5a)\) holds; see Exercise 16.1. \( \square \)

**Remark 16.9 (Well-posedness of discrete adjoint problem).** Consider the adjoint operator \( A_h^* : W_h \to V'_h \) defined in \((16.10)\). Then, \( A_h \) is an isomorphism if and only if \( A_h^* \) is an isomorphism; see Exercise 16.2. In other words, the well-posedness of the discrete problem \((16.2)\) is equivalent to the well-posedness of the following adjoint problem:

\[
\left\{ \begin{array}{l}
\text{Seek } y_h \in W_h \text{ such that } \\
a_h(v_h, y_h) = g_h(v_h), \quad \forall v_h \in V_h,
\end{array} \right.
\]
for any \( g_h \in V'_h \). Moreover, owing to Corollary A.61 (and since \( V_h \) is reflexive, being finite-dimensional), \( A_h \) and \( A_h^* \) satisfy the inf-sup condition \((16.5a)\) with the same constant \( \alpha_h \), i.e.,
\[
\inf_{v_h \in V_h} \sup_{w_h \in W_h} \frac{\langle A_h v_h, w_h \rangle_{W'_h, W_h}}{\| v_h \|_{V_h} \| w_h \|_{W_h}} = \inf_{w_h \in W_h} \sup_{v_h \in V_h} \frac{\langle A_h v_h, w_h \rangle_{W'_h, W_h}}{\| v_h \|_{V_h} \| w_h \|_{W_h}}. \quad (16.11)
\]
(Note that the numerator of the right-hand side equals \( \langle A_h^* w_h, v_h \rangle_{V'_h, V_h} \).) \( \square \)
16.3 Fortin’s criterion

Consider a conforming approximation, \( V_h \subset V \) and \( W_h \subset W \), and assume that the discrete problem (16.2) is formulated using the same bilinear form as in the continuous problem, i.e., \( a_h = a \). Then, if \( a \) is coercive on \( V \) with coercivity parameter \( \alpha \), \( a \) also coercive on \( V_h \) with coercivity parameter \( \alpha_h = \alpha \), and the discrete problem (16.2) with \( W_h = V_h \) is automatically well-posed. In the more general case of the BNB Theorem 15.10, the situation is different since neither condition (bnb1) nor condition (bnb2) imply its discrete counterpart. This means that the inf-sup condition (16.5a) needs to be proved. A powerful tool to achieve this is due to Fortin [210]; see also Boffi et al. [55, Prop. 5.4.3].

**Lemma 16.10 (Fortin’s criterion).** Let \( V \) and \( W \) be two real Banach spaces and let \( a \in \mathcal{B}(V,W) \). Assume that \( a \) satisfies the inf-sup condition (bnb1) with constant \( \gamma > 0 \). Let \( V_h \subset V \) and let \( W_h \subset W \) be equipped with the norms of \( V \) and \( W \), respectively. Consider the following two statements:

(i) There exists a mapping \( \Pi_h : W \to W_h \) and a real number \( \gamma_H > 0 \) such that \( a(v_h, \Pi_h w - w) = 0 \), for all \( (v_h, w) \in V_h \times W \), and \( \gamma_H \| \Pi_h w \|_W \leq \| w \|_W \) for all \( w \in W \).

(ii) \( a \) satisfies the inf-sup condition

\[
\inf_{v_h \in V_h} \sup_{w_h \in W_h} \frac{a(v_h, w_h)}{\| w_h \|_W} \geq \beta > 0,
\]

(16.12)

In the complex case, \( a(v_h, w_h) \) is replaced by \( \Re(a(v_h, w_h)) \) in (16.12).

Then, (i) \( \Rightarrow \) (ii) with \( \beta = \gamma H \alpha \) and (ii) \( \Rightarrow \) (i) with \( \gamma_H = \frac{\beta}{\| a \|_{\mathcal{B}(V,W)}} \).

**Proof.** We give the proof in the real case; the complex case is treated similarly.

1. Assuming (i), we have

\[
\sup_{w_h \in W_h} \frac{a(v_h, w_h)}{\| w_h \|_W} \geq \sup_{w \in W} \frac{a(v_h, \Pi_h w)}{\| \Pi_h w \|_W} = \sup_{w \in W} \frac{a(v_h, w)}{\| \Pi_h w \|_W} \geq \gamma_H \sup_{w \in W} \frac{a(v_h, w)}{\| w \|_W} \geq \gamma_H \alpha \| v_h \|_V,
\]

since \( a \) satisfies (bnb1) and \( V_h \subset V \). This proves (16.12) with \( \beta = \gamma_H \alpha \).

2. Conversely, assume that \( a \) satisfies (16.12). Let \( A_h : V_h \to W_h \) be the operator defined in (16.7), i.e., \( A_h(v_h, w_h)_{V_h^*, W_h^*} = a(v_h, w_h) \). We now apply the converse of Lemma A.57 with \( W_h \) in lieu of \( V \), \( V_h^* \) in lieu of \( W \), and \( A_h^* \) in lieu of \( A \); note that we identify \( V_h \) and \( V_h'' \), \( W_h \) and \( W_h'' \), since \( V_h \) and \( W_h \) are finite-dimensional. According to Lemma A.57, (16.12) implies that for all \( \ell_h \in V_h' \), there exists \( \Phi_h(\ell_h) \in W_h \) such that \( A_h^* \Phi_h(\ell_h) = \ell_h \) and \( \beta \| \Phi_h(\ell_h) \|_{W_h} \leq \| \ell_h \|_{V_h'} \). Let \( w \in W \) and define \( \ell_h(w) \in V_h' \) by setting \( \langle \ell_h(w), v_h \rangle_{V_h^*, V_h} := a(v_h, w) \) for all \( v_h \in V_h \). Let us define \( \Pi_h(w) = \Phi_h(\ell_h(w)) \).
Proof. \[a(v_h, \Pi_h(w)) = \langle A_h v_h, \Phi_h(\ell_h(w)) \rangle_{W_h^\prime, W_h} = \langle A_h^\prime \Phi_h(\ell_h(w)), v_h \rangle_{V_h^\prime, V_h} = \langle \ell_h(w), v_h, \rangle_{V_h^\prime, V_h} = a(v_h, w),\]
which proves that \(a(v_h, \Pi_h(w) - w) = 0\) for all \(w \in W\). Moreover
\[
\beta \|\Pi_h(w)\|_{W_h} = \beta \|\Phi_h(\ell_h(w))\|_{W_h} \leq \|\ell_h(w)\|_{V_h^\prime} \leq \|a\|_{B(V,W)} \|w\|_W,
\]
which proves that \(\frac{\beta}{\|a\|_{B(V,W)}} \|\Pi_h(w)\|_{W_h} = \frac{\beta}{\|a\|_{B(V,W)}} \|\Pi_h(w)\|_W \leq \|w\|_W\). \(\square\)

**Remark 16.11** (Equivalence, linearity). Only the forward implication in Fortin’s criterion is usually found in the literature. Note that there is a gap in the stability constant \(\gamma_H\) between the direct and converse statements, since the ratio of the two is equal to \(\frac{\|a\|_{B(V,W)}}{\alpha}\) (which is independent of the discrete setting). Moreover, in the converse statement, the mapping \(\Pi_h\) can be nonlinear (since the mapping \(\Phi_h\) can be nonlinear). The existence of a linear mapping is granted in a Hilbertian setting, see Remark A.58. \(\square\)

### 16.4 Basic error estimate

In this section, we bound the approximation error \((u - u_h)\). We consider the relatively simple setting of a conforming approximation \((V_h \subset V\) and \(W_h \subset W\)) with \(a_h = a\) and \(\ell_h = \ell\). We assume that the inf-sup condition \((16.5a)\) holds, as well as \(\dim(V_h) = \dim(W_h)\), so that the discrete problem \((16.2)\) is well-posed. Simple sufficient conditions for \((16.5a)\) to hold are \(W_h = V_h\) and \(a\) coercive on \(V_h\) (for which a sufficient condition is that \(a\) be coercive on \(V\)).

**Lemma 16.12 (Galerkin orthogonality).** Let \(u\) solve \((16.1)\) and let \(u_h\) solve \((16.2)\) with \(V_h \subset V\), \(W_h \subset W\), \(a_h = a\), and \(\ell_h = \ell\). Then, the following holds:
\[
a(u - u_h, w_h) = 0, \quad \forall w_h \in W_h. \tag{16.13}
\]

*Proof.* For all \(w_h \in W_h\), we observe that \(a(u, w_h) = \ell(w_h)\) since \(W_h \subset W\) and \(\ell(w_h) = a(u_h, w_h)\) since \(a_h = a\) and \(\ell_h = \ell\). \(\square\)

The starting point of the error analysis is to make sure that \((16.2)\) is consistent with the original problem \((16.1)\). Loosely speaking, consistency is checked by inserting the exact solution into the discrete problem and verifying that the result is small. In the present situation, we infer that \(a(u, w_h) = \ell(w_h)\) for all \(w_h \in W_h\) (since \(a_h = a\), \(\ell_h = \ell\), and \(W_h \subset W\)), i.e., the approximate problem is exactly consistent with the continuous one. This exact consistency property is just a reformulation of the Galerkin orthogonality.

Let us now bound the approximation error. Since the approximation setting is conforming, we equip the subspaces \(V_h \subset V\) and \(W_h \subset W\) with the norms \(\|\cdot\|_V\) and \(\|\cdot\|_W\), respectively. Recall that \(\|a\|_{B(V,W)} := \sup_{0 \neq (v,w) \in V \times W} \frac{a(v,w)}{\|v\|_V \|w\|_W}\).
Lemma 16.13 (Basic error estimate). Assume that problems (16.1) and (16.2) are well-posed with solutions \( u \) and \( u_h \). Assume that the approximation setting is conforming with \( a_h = a \) and \( \ell_h = \ell \). Let \( \alpha_h > 0 \) be the constant in the inf-sup condition (16.5a). Then, the following holds:

\[
\|u - u_h\|_V \leq \left(1 + \frac{\|a\|_{B(V,W)}}{\alpha_h}\right) \inf_{v_h \in V_h} \|u - v_h\|_V. \tag{16.14}
\]

Consequently, if approximability holds in the sense that

\[
\lim_{h \to 0} \inf_{v_h \in V_h} \|v - v_h\|_V = 0, \quad \forall v \in V, \tag{16.15}
\]

then \( \lim_{h \to 0} \|u - u_h\|_V = 0 \).

Proof. Let \( v_h \in V_h \). Using stability (i.e., the inf-sup condition (16.5a)), consistency (i.e., Galerkin orthogonality), and the boundedness of \( a \) yields

\[
\alpha_h \|u_h - v_h\|_V \leq \sup_{w_h \in W_h} \frac{a(u_h - v_h, w_h)}{\|w_h\|_W} = \sup_{w_h \in W_h} \frac{a(u - v_h, w_h)}{\|w_h\|_W} \leq \|a\|_{B(V,W)} \|u - v_h\|_V.
\]

We conclude using the triangle inequality. \( \square \)

Lemma 16.13 is derived in Babuška and Aziz [29]. Note that (16.14) implies that if the exact solution turns out to be in \( V_h \), then \( u_h = u \). Whenever \( V \) and \( W \) are Hilbert spaces, the error estimate can be sharpened as follows:

\[
\|u - u_h\|_V \leq \left(\frac{\|a\|_{B(V,W)}}{\alpha_h}\right) \inf_{v_h \in V_h} \|u - v_h\|_V, \tag{16.16}
\]

see Xu and Zikatanov [435] and Exercise 16.7. Moreover, if the bilinear form \( a \) is coercive (with parameter \( \alpha \)), \( W = V \), and \( W_h = V_h \), the error estimate is known as Céa’s Lemma [125] and takes the form (see Exercise 16.6)

\[
\|u - u_h\|_V \leq \left(\frac{\|a\|_{B(V,V)}}{\alpha}\right) \inf_{v_h \in V_h} \|u - v_h\|_V. \tag{16.17}
\]

Exercises

Exercise 16.1 (Condition (b\(n\)2)). Prove that (16.9) is equivalent to (16.5b) provided (16.5a) holds. (Hint: use that \( \dim(W_h) = \text{rank}(A_h) + \dim(\ker(A_h^*)) \), with \( A_h^* \) defined in (16.10), together with the Rank Theorem.)

Exercise 16.2 (Bijectivity of \( A_h^* \)). Prove that \( A_h \) is an isomorphism if and only if \( A_h^* \) is an isomorphism. (Hint: use \( \dim(V_h) = \text{rank}(A_h^*) + \dim(\ker(A_h^*)) \) and \( \dim(W_h) = \text{rank}(A_h) + \dim(\ker(A_h^*)) \).)
Exercise 16.3 (Minimal residual, Petrov–Galerkin). Let \( V, W \) be two Hilbert spaces, let \( A \in \mathcal{L}(V; W') \) be an isomorphism, and let \( \ell \in W' \). Consider a finite-dimensional subspace \( V_h \subset V \). We want to approximate the unique solution \( u \in V \) satisfying \( Au = \ell \) by using a conforming Petrov–Galerkin approximation with \( V_h = \text{span} \{ v \} \) where \( J_W : W \to W' \) is the Riesz–Fréchet isomorphism. The discrete bilinear form is \( a_h(v_h, w_h) = (Av_h, w_h)_{W', W} \), and the discrete linear form is \( \ell_h(w_h) = \ell(w_h) \).

(i) Prove that the discrete problem (16.2) is well-posed.

(ii) Show that its unique solution minimizes the residual functional \( R(v) = \| Av - \ell \|_{W'} \), over \( V_h \).

Exercise 16.4 (Variant of Fortin's criterion). Write a variant of Fortin’s criterion assuming \( V, W \) reflexive and using this time an operator \( \Pi_h : V \to V_h \) such that \( a(\Pi_h v - v, w_h) = 0 \) for all \( (v, w_h) \in V \times W_h \) and \( \gamma_H \| \Pi_h v \|_V \leq \| v \|_V \) for all \( v \in V \) for some \( \gamma_H > 0 \). (Hint: use (16.11) and Corollary A.61.)

Exercise 16.5 (Fortin's criterion). Let \( V \) and \( W \) be two real Banach spaces and let \( a \in \mathcal{B}(V, W) \). Let \( V_h \subset V \) and \( W_h \subset W \) be equipped with the norms of \( V \) and \( W \), respectively. Assume that there are two operators (not necessarily linear) \( \Pi_{1,h}, \Pi_{2,h} : W \to W_h \) and two uniform constants \( c_1, c_2 > 0 \) such that \( \| \Pi_{1,h} w \|_W \leq c_1 \| w \|_W \), \( \| \Pi_{2,h}(I - \Pi_{1,h})w \|_W \leq c_2 \| w \|_W \) and \( a(v_h, \Pi_{2,h} w) = a(v_h, \Pi_{2,h} w) \) for all \( v_h \in V_h, w \in W \). Prove that \( \Pi_h = \Pi_{1,h} + \Pi_{2,h}(I - \Pi_{1,h}) \) satisfies Fortin’s criterion.

Exercise 16.6 (Céa’s Lemma). Assume that \( a \) is coercive on \( V \) with parameter \( \alpha \). Take \( W = V \) and \( W_h = V_h \). Assume \( V_h \subset V \) and \( a_h = a \). Prove (16.17) and \( \| u - u_h \|_V \leq \left( \frac{\| a \|_{\mathcal{L}(V, W)}}{\alpha} \right)^{\frac{1}{2}} \inf_{v_h \in V_h} \| u - v_h \|_V \) if \( a \) is symmetric.

Exercise 16.7 (Xu–Zikatanov Lemma). Let \( V, W \) be two Hilbert spaces, let \( V_h \subset V \) and \( W_h \subset W \) with \( \dim(V_h) = \dim(W_h) \), and let \( a \) be a bounded bilinear form on \( V \times W \) satisfying the stability property (??). Let \( P_h : V \to V_h \) map \( u \in V \) to the unique solution of \( a(u - u_h, w_h) = 0 \) for all \( w_h \in W_h \).

(i) Prove that \( P_h \in \mathcal{L}(V; V) \) with \( \| P_h \|_{\mathcal{L}(V; V)} \leq \frac{\| a \|_{\mathcal{L}(V, W)}}{\alpha} \).

(ii) Prove that \( P_h \circ P_h = P_h \) and that \( (I - P_h)(v_h) = 0 \) for all \( v_h \in V_h \).

(iii) Prove (16.16). (Hint: see the proof of Theorem 1.11.)
Solution to exercises

Exercise 16.1 (Condition (\text{bnb2})). The statement (16.9) is equivalent to \( \ker(A_h^*) = \{0\} \). Since \( \dim(W_h) = \rank(A_h) + \dim(\ker(A_h^*)) \), this statement is equivalent to \( \dim(W_h) = \rank(A_h) \). Since the inf-sup condition (16.5a) implies that \( \ker(A_h^*) = \{0\} \) so that \( \rank(A_h) = \dim(V_h) \) owing to the Rank Theorem, we conclude that (16.9) is equivalent to \( \dim(V_h) = \dim(W_h) \).

Exercise 16.2 (Bijectivity of \( A_h^* \)). We observe that \( A_h^*: V_h \to W_h' \) is an isomorphism if and only if \( \ker(A_h) = \{0\} \) and \( \rank(A_h) = \dim(W_h') = \dim(W_h) \). Since \( \dim(V_h) = \rank(A_h^*) + \dim(\ker(A_h)) \) and \( \dim(W_h) = \rank(A_h) + \dim(\ker(A_h^*)) \), these two statements are equivalent to \( \dim(V_h) = \rank(A_h^*) \) and \( \dim(\ker(A_h^*)) = 0 \), i.e., to the bijectivity of \( A_h^* \).

Exercise 16.3 (Minimal residual, Petrov–Galerkin). (i) We apply the discrete BNB Theorem 16.6. Since \( J^{-1}_W A: V \to W \) is an isomorphism, the subspaces \( V_h \) and \( W_h \) have the same dimension, thus proving (16.5b). Moreover, for all \( v_h \in V_h \), taking \( w_h = J^{-1}_W A v_h \in W_h \), we observe that
\[
a_h(v_h, w_h) = \langle Av_h, J^{-1}_W A v_h \rangle_{W', W} = \|Av_h\|_{W'}^2 \geq \alpha \|v_h\|_V^2,
\]
for some real number \( \alpha > 0 \) since \( A: V \to W' \) is an isomorphism. Moreover, \( \|w_h\|_W = \|Av_h\|_{W'} \leq \|A\|_{L(V; W')} \|v_h\|_V \). Combining these two bounds yields the inf-sup condition (16.5a).

(ii) We observe that
\[
\Re(v)^2 = \langle Av - \ell, J^{-1}_W (Av - \ell) \rangle_{W', W}, \quad \forall v \in V.
\]
Proceeding as in the proof of Proposition 15.8, we then infer that \( u_h \) minimizes \( \Re \) over \( V_h \) if and only if
\[
\langle Au_h - \ell, J^{-1}_W A v_h \rangle_{W', W} = 0, \quad \forall v_h \in V_h,
\]
which is just a reformulation of the discrete problem in the Petrov–Galerkin setting.

Exercise 16.4 (Fortin’s criterion). Let \( V \) and \( W \) be two real Banach spaces and let \( a \in B(V, W) \). Assume that \( a \) satisfies the inf-sup condition (\text{bnb1}) with constant \( \alpha > 0 \). Let \( V_h \subset V \) and let \( W_h \subset W \) be equipped with the norms of \( V \) and \( W \), respectively. Consider the following two statements:

(i) There exists a mapping \( \Pi_h: V \to V_h \) and a real number \( \gamma_H > 0 \) such that \( a(\Pi_h v - v_h, w_h) = 0 \), for all \( (v, w_h) \in V \times W_h \), and \( \gamma_H \|\Pi_h v\|_V \leq \|v\|_V \) for all \( v \in V \).

(ii) \( a \) satisfies the inf-sup condition (16.12).
Let us prove the following two statements: (i) \( \Rightarrow \) (ii) with \( \beta = \gamma_H \alpha \) and (ii) \( \Rightarrow \) (i) with \( \gamma_H = \frac{\beta}{\|a\|_{B(V,W)}} \).

(1) The direct statement. we observe that, for all \( w_h \in W_h \),
\[
\sup_{v_h \in V_h} \frac{a(v_h, w_h)}{\|v_h\|_V} \geq \sup_{v \in V} \frac{a(I_{1,h}v, w_h)}{\|I_{1,h}v\|_V} \geq \gamma_H \sup_{v \in V} \frac{a(v, w_h)}{\|v\|_V}.
\]

Owing to Corollary A.61 (with \( W' \) in lieu of \( W \)) and since \( V \) is reflexive, we infer that
\[
\alpha = \inf_{v \in V} \sup_{w \in W} \frac{a(v, w)}{\|w\|_W \|v\|_V} = \inf_{w \in W'} \sup_{v \in V} \frac{a(v, w)}{\|w\|_W \|v\|_V},
\]
where we have used the reflexivity of \( W \) and the fact that \( a(v, w) = \langle Av, w \rangle_W \). Since \( w_h \in W_h \subset W \), this implies that \( \sup_{v \in V} \frac{a(v, w_h)}{\|v\|_V} \geq \alpha \|w_h\|_W \), thereby proving that
\[
\inf_{w_h \in W_h} \sup_{v_h \in V_h} \frac{a(v_h, w_h)}{\|v_h\|_V \|w_h\|_W} \geq \gamma_H \alpha.
\]

Upon invoking (16.11) we conclude that \( \alpha \) satisfies the inf-sup condition (16.12) with \( \beta = \gamma_H \alpha \).

(2) The converse statement. Proceeding as in the proof of Lemma 16.10, this time using that \( \inf_{v_h \in W_h} \sup_{v_h \in V_h} \frac{a(v_h, w_h)}{\|v_h\|_V \|w_h\|_W} = \beta \) owing to (16.11), we infer that for all \( \ell_h \in W' \), there exists \( \Phi_h(\ell_h) \in V_h \) such that \( A_h \Phi_h(\ell_h) = \ell_h \) and \( \beta \|\Phi_h(\ell_h)\|_{V_h} \leq \|\ell_h\|_{W'_h} \). Still proceeding as in the proof of Lemma 16.10, one readily verifies that the mapping \( \Pi_h : V \rightarrow V_h \) such that \( \Pi_h(v) = \Phi_h(\ell_h(v)) \) for all \( v \in V \), verifies the requirements of statement (i), where \( \ell_h(v) \in W'_h \) is defined such that \( (\ell_h(v), w_h)_{W'_h,V_h} = a(v, w_h) \).

Exercise 16.5 (Fortin’s criterion). Let \( V \) and \( W \) be two real Banach spaces and let \( a \in B(V,W) \). Let \( V_h \subset V \) and let \( W_h \subset W \) be equipped with the norms of \( V \) and \( W \), respectively. Assume that there are two operators (not necessarily linear) \( \Pi_{1,h}, \Pi_{2,h} : W \rightarrow W_h \) and two uniform constants \( c_1, c_2 > 0 \) such that \( \|\Pi_{1,h}w\|_W \leq c_1\|w\|_W \), \( \|\Pi_{2,h}(I - \Pi_{1,h})w\|_W \leq c_2\|w\|_W \) and \( a(v_h, \Pi_{2,h}w) = a(v_h, \Pi_{2,h}(I - \Pi_{1,h})w) \) for all \( v_h \in V_h, w \in W \). Let \( \Pi_h w = \Pi_{1,h}w + \Pi_{2,h}(I - \Pi_{1,h})w \) for all \( w \in W \). Then
\[
a(v_h, \Pi_h w) = a(v_h, \Pi_{1,h}w + \Pi_{2,h}(I - \Pi_{1,h})w) = a(v_h, \Pi_{1,h}w) + a(v_h, \Pi_{2,h}(I - \Pi_{1,h})w) = a(v_h, \Pi_{1,h}w) + a(v_h, (I - \Pi_{1,h})w) = a(v_h, w).
\]

Moreover,
\[
\|\Pi_h w\|_W \leq \|\Pi_{1,h}w\|_W + \|\Pi_{2,h}(I - \Pi_{1,h})w\|_W \leq (c_1 + c_2)\|w\|_W.
\]
Hence \( \Pi_h \) satisfies Fortin’s criterion.
Exercise 16.6 (Céa’s Lemma). Exploiting the coercivity of $a$ on $V$ and then using the Galerkin orthogonality and the boundedness of $a$ on $V \times V$ yields

$$\alpha \|u - u_h\|_V^2 \leq a(u - u_h, u - u_h) = a(u - u_h, u - v_h) \leq \|a\|_{B(V,V)} \|u - u_h\|_V \|u - v_h\|_V, \quad \forall v_h \in V_h,$$

whence we infer (16.17). Moreover, if $a$ is symmetric, the Galerkin orthogonality can be used twice to infer that

$$\alpha \|u - u_h\|_V^2 \leq a(u - u_h, u - u_h) = a(u - v_h, u - v_h) \leq \|a\|_{B(V,V)} \|u - v_h\|_V^2.$$

Exercise 16.7 (Xu–Zikatanov Lemma).

(i) Given $u \in V$, $a(u, w_h)$ defines a bounded linear form on $W_h$. Stability and $\dim(V_h) = \dim(W_h)$ imply that $u_h \in V_h$ is uniquely defined. The linearity of $P_h$ is obvious. Moreover,

$$\alpha_h \|P_h(u)\|_V \leq \sup_{w_h \in W_h} \frac{a(P_h(u), w_h)}{\|w_h\|_W} = \sup_{w_h \in W_h} \frac{a(u, w_h)}{\|w_h\|_W} \leq \|a\|_{B(V,W)} \|u\|_V,$$

whence the bound on $\|P_h\|_{\mathcal{L}(V;V)}$.

(ii) If $u \in V_h$, uniqueness of the solution to the discrete problem implies that $u_h = P_h(u) = u$. Since $P_h(u) \in V_h$, we infer that $P_h(P_h(u)) = P_h(u)$. Moreover, letting $v_h \in V_h$, we obtain $v_h = P_h(v_h)$.

(iii) Since $u - u_h = (I - P_h)(u) = (I - P_h)(u - v_h)$ for all $v_h \in V_h$, we infer that

$$\|u - u_h\|_V \leq \|I - P\|_{\mathcal{L}(V;V)} \|u - v_h\|_V = \|P\|_{\mathcal{L}(V;V)} \|u - v_h\|_V,$$

using the hint, and we conclude using the above bound on $\|P_h\|_{\mathcal{L}(V;V)}$. 