The goal of this chapter is to construct $\mathbb{R}^d$-valued finite elements $(K, P, \Sigma)$ with $d \geq 2$ such that (i) $\mathcal{P}_{k, d} := [\mathcal{P}_{k, d}]^d \subset P$ for some $k \geq 0$ and (ii) the degrees of freedom (dofs) in $\Sigma$ fully determine the normal components of the polynomials in $P$ on all the faces of $K$. The first requirement is key for proving convergence rates on the interpolation error. The second one is key for constructing $H(\text{div})$-conforming finite element spaces (see Chapter 19). The finite elements introduced in this chapter are used, e.g., in Chapter 50 to approximate Darcy’s equations which constitute a fundamental model for porous media flows. The focus here is on defining a reference element and generating finite elements on the mesh cells. The estimation of the interpolation error is done in Chapters 16 and 17. We detail the construction for the simplicial Raviart–Thomas finite elements. Some alternative elements are outlined at the end of the chapter.

14.1 The lowest-order case

We start by considering the lowest-order Raviart–Thomas finite element. Let $d \geq 2$ be the space dimension, and define the polynomial space

$$\mathbf{RT}_{0,d} := \mathcal{P}_{0,d} \oplus x \mathcal{P}_{0,d}. \tag{14.1}$$

Since the above sum is indeed direct, $\mathbf{RT}_{0,d}$ is a vector space of dimension $\dim(\mathbf{RT}_{0,d}) = d + 1$. A basis of $\mathbf{RT}_{0,2}$ is $\{(1, 0), (0, 1), (x_1, x_2)\}$. The space $\mathbf{RT}_{0,d}$ has several further interesting properties. (a) One has $\mathcal{P}_{0,d} \subset \mathbf{RT}_{0,d}$, in agreement with the first requirement stated above. (b) If $v \in \mathbf{RT}_{0,d}$ is divergence-free, then $v$ is constant. (c) If $H$ is an affine hyperplane of $\mathbb{R}^d$ with normal vector $\nu_H$, then the function $v \cdot \nu_H$ is constant on $H$ for all $v \in \mathbf{RT}_{0,d}$. Writing $v(x) = a + bx$ with $a \in \mathbb{R}^d$ and $b \in \mathbb{R}$, we indeed have $(v(x_1) - v(x_2)) \cdot \nu_H = b(x_1 - x_2) \cdot \nu_H = 0$ for all $x_1, x_2 \in H$. 
Let $K$ be a simplex in $\mathbb{R}^d$ and let $F_K$ be the collection of the faces of $K$. Any face $F \in F_K$ is oriented by a fixed unit normal vector $n_F$, and we set $\nu_F := |F| n_F$. Let $\Sigma$ be the collection of the following linear forms acting on $\mathbb{R}^n$:

$$
\sigma^F_F(v) := \frac{1}{|F|} \int_F (v \cdot \nu_F) \, ds, \quad \forall F \in F_K.
$$

(14.2)

Since $v \cdot \nu_F$ is constant on $F$, $\sigma^F_F(v) = 0$ implies that $v \cdot \nu_F = 0$, in agreement with the second requirement stated above. Note that we could have written more simply $\sigma^F_F(v) = \int_F (v \cdot n_F) \, ds$, but the expression (14.2) is introduced to be consistent with later notation. In any case, the unit of $\sigma^F_F(v)$ is a surface times the dimension of $v$. A graphic representation of the dofs is shown in Figure 14.1.

**Proposition 14.1 (Finite element).** $(K, \mathbb{R}^n, \Sigma)$ is a finite element.

**Proof.** Since $\dim(\mathbb{R}^n) = \dim(\Sigma) = d + 1$, we just need to prove that the only function $v \in \mathbb{R}^n$ that annihilates the dofs in $\Sigma$ is zero. Since $v \cdot \nu_F$ is constant and has zero mean-value on $F$, we have $v_F \cdot \nu_F = 0$ for all $F \in F_K$. Moreover the divergence theorem implies that $\int_K (\nabla \cdot v) \, dx = \sum_{F \in F_K} \int_F (v \cdot n_F) \, ds = 0$. Since $\nabla \cdot v \in \mathbb{P}_0$, we infer that $\nabla \cdot v$ is zero, so that $v \in \mathbb{P}_0$. Hence $v \cdot \nu_F$ vanishes identically in $K$ for all $F \in F_K$. Since $\dim(\nu_F) = \mathbb{R}^d$ (see Exercise 7.2(iv)), we have $v = 0$. \qed

Using the fact that the volume of a simplex is $|K| = \frac{1}{d} |F| h_F^K$ for all $F \in F_K$ where $h_F^K$ is the height of $K$ measured from the vertex $z_F$ opposite to $F$, one readily verifies that the shape functions are

$$
\theta^F_F(x) := \frac{i_{F,K}}{d |K|} (x - z_F), \quad \forall x \in \mathbb{R}^d, \forall F \in F_K,
$$

(14.3)

where $i_{F,K} := 1$ if $\nu_F$ points outward $K$ and $i_{F,K} := -1$ otherwise (i.e., $i_{F,K} = n_F \cdot n_K$ where $n_K$ is the outward unit normal to $K$). The normal component of $\theta^F_F$ is constant on each of the $(d+1)$ faces of $K$ (as expected),
it is equal to 1 on $F$ and to 0 on the other faces. See Exercise 14.1 for additional properties of the $\mathbf{RT}_{0,d}$ shape functions.

### 14.2 The polynomial space $\mathbf{RT}_{k,d}$

We now generalize the construction of §14.1 to arbitrary polynomial order $k \in \mathbb{N}$. Let $d \geq 2$ be the space dimension. Recall from §7.3 the multi-index set $\mathcal{A}_{k,d} := \{ \alpha \in \mathbb{N}^d \mid |\alpha| \leq k \}$ where $|\alpha| := \alpha_1 + \ldots + \alpha_d$. We additionally introduce the subset $\mathcal{A}_{k,d}^H := \{ \alpha \in \mathcal{A}_{k,d} \mid |\alpha| = k \}$. For instance, $\mathcal{A}_{1,2} = \{ (0,0), (1,0), (0,1) \}$ and $\mathcal{A}_{1,2}^H = \{ (1,0), (0,1) \}$.

**Definition 14.2 (Homogeneous polynomials).** A polynomial $p \in \mathbb{P}_{k,d}$ is said to be homogeneous of degree $k$ if $p(x) = \sum_{\alpha \in \mathcal{A}_{k,d}^H} a_\alpha x_\alpha$ with real coefficients $a_\alpha$. The real vector space composed of homogeneous polynomials is denoted $\mathbb{P}_{k,d}^H$ or $\mathbb{P}_k^H$ when the context is unambiguous.

**Lemma 14.3 (Properties of $\mathbb{P}_{k,d}^H$).** We have $x \cdot \nabla q = kq$ (Euler’s identity) and $\nabla \cdot (qx) = (k + d)q$ for all $q \in \mathbb{P}_{k,d}^H$.

**Proof.** By linearity it suffices to verify the assertion with $q(x) := x_\alpha$ for all $\alpha \in \mathcal{A}_{k,d}^H$. We have $x \cdot \nabla q = \sum_{i \in \{1 : d\}} \alpha_i x_i x_\alpha^i = (\sum_{i \in \{1 : d\}} \alpha_i) q = kq$. Moreover the assertion for $\nabla \cdot (qx)$ follows from the observation that $\nabla \cdot x = d$ and $\nabla \cdot (qx) = q \nabla \cdot x + x \cdot \nabla q$. □

**Definition 14.4 ($\mathbf{RT}_{k,d}$).** Let $k \in \mathbb{N}$ and let $d \geq 2$. We define the following real vector space of $\mathbb{R}$-valued polynomials:

$$\mathbf{RT}_{k,d} := \mathbb{P}_{k,d} \oplus x \mathbb{P}_{k,d}^H.$$

(14.4)

The above sum is direct since polynomials in $x \mathbb{P}_{k,d}^H$ are members of $\mathbb{P}_{k+1,d}^H$, whereas the degree of any polynomial in $\mathbb{P}_{k,d}$ does not exceed $k$.

**Example 14.5** ($k = 1, d = 2$). $\dim(\mathbf{RT}_{1,2}) = 8$ and $\{ (1,0), (2,0), (0,1), (0,0), (x_1), (x_2), (x_1^2), (x_2^2) \}$ is a basis of $\mathbf{RT}_{1,2}$.

**Lemma 14.6 (Dimension of $\mathbf{RT}_{k,d}$).** $\dim(\mathbf{RT}_{k,d}) = (k+d+1)(k+d-1)$, in particular $\dim(\mathbf{RT}_{k,2}) = (k+1)(k+3)$ and $\dim(\mathbf{RT}_{k,3}) = \frac{1}{2}(k+1)(k+2)(k+4)$.

**Proof.** Since $\dim(\mathbb{P}_{k,d}) = \binom{k+d}{k}$, $\dim(\mathbb{P}_{k,d}^H) = \binom{k+d-1}{k}$, and the sum in (14.4) is direct, $\dim(\mathbf{RT}_{k,d}) = d \binom{k+d}{k} + \binom{k+d-1}{k} = (k + d + 1) \binom{k+d-1}{k}$. □

**Lemma 14.7 (Trace space).** Let $H$ be an affine hyperplane in $\mathbb{R}^d$ with normal vector $\mathbf{n}_H$, and let $T_H : \mathbb{R}^{d-1} \to H$ be an affine bijective mapping. Then $\mathbf{v}_H \cdot \mathbf{n}_H \in \mathbb{P}_{k,d-1} \circ T_H^{-1}$ for all $\mathbf{v} \in \mathbf{RT}_{k,d}$. 
Proof. Let \( v \in \mathbf{RT}_{k,d} \) with \( v = p + xq \), \( p \in \mathbb{P}_{k,d} \), and \( q \in \mathbb{P}_h^{k,d} \). Let \( x \in H \) and set \( y = T_H^{-1}(x) \). Since the quantity \( x \cdot n_H \) is constant, say \( x \cdot n_H = c_H \), we infer that \((v |_{T_H} \cdot n_H) (x) = (p |_{T_H} \cdot n_H) (x) + (x \cdot n_H) q(x) = ((p \circ T_H) \cdot n_H)(y) + c_H(q \circ T_H)(y) \), and both terms in the sum are in \( \mathbb{P}_{k,d-1} \). \( \blacklozenge \)

Remark 14.8 (\( T_H \)). Consider a second affine bijective mapping \( \tilde{T}_H : \mathbb{R}^{d-1} \rightarrow H \). Since \( S := T_H^{-1} \circ \tilde{T}_H \) is an affine bijective mapping from \( \mathbb{R}^{d-1} \) onto itself, we have \( \mathbb{P}_{k,d-1} \circ S = \mathbb{P}_{k,d-1} \). Hence \( \mathbb{P}_{k,d-1} \circ T_H^{-1} = \mathbb{P}_{k,d-1} \circ S \circ \tilde{T}_H^{-1} = \mathbb{P}_{k,d-1} \circ \tilde{T}_H^{-1} \). This proves that the assertion of Lemma 14.7 is independent of the mapping \( T_H \). \( \blacklozenge \)

Lemma 14.9 (Divergence). \( \nabla \cdot v \in \mathbb{P}_{k,d} \) for all \( v \in \mathbf{RT}_{k,d} \), and if the function \( v \) is divergence-free, then \( v \in \mathbb{P}_{k,d} \).

Proof. That \( \nabla \cdot v \in \mathbb{P}_{k,d} \) follows from \( v_i \in \mathbb{P}_{k+1,d} \) for all \( i \in \{1:d\} \). Let \( v \in \mathbf{RT}_{k,d} \) be divergence-free. Since \( v \in \mathbf{RT}_{k,d} \), there are \( p \in \mathbb{P}_{k,d} \) and \( q \in \mathbb{P}_h^{k,d} \) such that \( v = p + xq \). Owing to Lemma 14.3, we infer that \( \nabla \cdot p + (k+d)q = 0 \), which implies that \( q = 0 \) since \( \mathbb{P}_{k,d} \cap \mathbb{P}_{k-1,d} = \{0\} \) if \( k \geq 1 \), the argument for \( k = 0 \) being trivial. Hence, \( v = p \in \mathbb{P}_{k,d} \). \( \blacklozenge \)

14.3 Simplicial Raviart–Thomas elements

Let \( k \in \mathbb{N} \) and let \( d \geq 2 \). Let \( K \) be a simplex in \( \mathbb{R}^d \). Each face \( F \in \mathcal{F}_K \) of \( K \) is oriented by the normal vector \( \boldsymbol{v}_F := |F| \boldsymbol{n}_F \) (so that \( ||\boldsymbol{v}_F||_2 = |F| \)). Moreover, the simplex \( K \) itself is oriented by the \( d \) vectors \( \{ \boldsymbol{v}_{K,j} := |F_j| \boldsymbol{n}_{F_j} \}_{j \in \{1:d\}} \) where \( \{F_j\}_{j \in \{1:d\}} \) are the \( d \) faces of \( K \) sharing the vertex with the lowest enumeration index. Note that \( \{ \boldsymbol{v}_{K,j} \}_{j \in \{1:d\}} \) is a basis of \( \mathbb{R}^d \) (see Exercise 7.2(iv)) and that it is the canonical Cartesian basis of \( \mathbb{R}^d \) when \( K \) is the unit simplex.

The dofs of the \( \mathbf{RT}_{k,d} \) finite element involve integrals over the faces of \( K \) or over \( K \) itself (for \( k \geq 1 \)). The face dofs require to evaluate moments against \((d-1)\)-variate polynomials. To this purpose, we introduce an affine bijective mapping \( T_F : \hat{S}^{d-1} \rightarrow F \) for all \( F \in \mathcal{F}_K \), where \( \hat{S}^{d-1} \) is the unit simplex of \( \mathbb{R}^{d-1} \); see Figure 14.2. For instance, after enumerating the \( d \) vertices of \( \hat{S}^{d-1} \) and the \((d+1)\) vertices of \( K \), we can define \( T_F \) such that the \( d \) vertices of \( \hat{S}^{d-1} \) are mapped to the \( d \) vertices of \( F \) with increasing indices.

![Fig. 14.2](image)

\( \text{Fig. 14.2} \) Reference face \( \hat{S}^{d-1} \) and mapping \( T_F \) for \( d = 2 \) (left, the face \( F \) is indicated in bold) and \( d = 3 \) (right, the face \( F \) is highlighted in grey).
Definition 14.10 (dofs). We denote by $\Sigma$ the collection of the following linear forms acting on $\mathbb{RT}_{k,d}$:

$$
\sigma^f_{F,m}(v) := \frac{1}{|F|} \int_F (v \cdot \nu_F)(\zeta_m \circ T_F^{-1}) \, ds, \quad \forall F \in \mathcal{F}_K, \quad (14.5a)
$$

$$
\sigma^e_{j,m}(v) := \frac{1}{|K|} \int_K (v \cdot \nu_{K,j}) \psi_m \, dx, \quad \forall j \in \{1:d\}, \quad (14.5b)
$$

where $\{\zeta_m\}_{m \in \{1:n^{sh}_k\}}$ is a basis of $\mathbb{P}_{k,d-1}$ with $n^{sh}_k := \dim(\mathbb{P}_{k,d-1}) = \binom{d+k-1}{k}$ and $\{\psi_m\}_{m \in \{1:n^{sh}_k\}}$ is a basis of $\mathbb{P}_{k-1,d}$ with $n^{sh}_k := \dim(\mathbb{P}_{k-1,d}) = \binom{d+k-1}{k}$ if $k \geq 1$. We reorganize the dofs as follows:

$$
\Sigma^f_F := \{\sigma^f_{F,m}\}_{m \in \{1:n^{sh}_k\}}, \quad \forall F \in \mathcal{F}_K, \quad (14.6a)
$$

$$
\Sigma^e := \{\sigma^e_{j,m}\}_{(j,m) \in \{1:d\} \times \{1:n^{sh}_k\}}, \quad (14.6b)
$$

Remark 14.11 (dofs). The unit of all the dofs is a surface times the dimension of $v$. We could also have written $\sigma^e_{j,m}(v) = \ell_{K}^{-1} \int_K (v \cdot e_j) \psi_m \, dx$ for the cell dofs, where $\ell_K$ is a length scale of $K$ and $\{e_j\}_{j \in \{1:d\}}$ is the canonical Cartesian basis of $\mathbb{R}^d$. We will see that the definition (14.5b) is more natural when using the contravariant Piola transformation to generate other finite elements. The dofs are defined here on $\mathbb{RT}_{k,d}$. Their extension to some larger space $V(K)$ is addressed in Chapters 16-17.

Lemma 14.12 (Invariance w.r.t. $T_F$). Assume that every affine bijective mapping $S: \hat{S}^{d-1} \rightarrow \hat{S}^{d-1}$ leaves the basis $\{\zeta_m\}_{m \in \{1:n^{sh}_k\}}$ globally invariant, i.e., $\{\zeta_m\}_{m \in \{1:n^{sh}_k\}} = \{\zeta_m \circ S\}_{m \in \{1:n^{sh}_k\}}$. Then for all $F \in \mathcal{F}_K$, the set $\Sigma^f_F$ is independent of the affine bijective mapping $T_F$.

Proof. Let $T_F, \hat{T}_F$ be two affine bijective mappings from $\hat{S}^{d-1}$ to $F$. Then $S := T_F^{-1} \circ \hat{T}_F$ is an affine bijective mapping from $\hat{S}^{d-1}$ to $\hat{S}^{d-1}$. Let $m \in \{1:n^{sh}_k\}$. The invariance assumption implies that there exists $\zeta_n, n \in \{1:n^{sh}_k\}$, s.t. $\zeta_m \circ S = \zeta_n$. Hence we have

$$
|F| \sigma^f_{F,m}(v) = \int_F (v \cdot \nu_F)(\zeta_m \circ T_F^{-1}) \, ds
$$

$$
= \int_F (v \cdot \nu_{F})((\zeta_m \circ S) \circ \hat{T}_F^{-1}) \, ds = \int_F (v \cdot \nu_F)(\zeta_n \circ \hat{T}_F^{-1}) \, ds. \quad \square
$$

Example 14.13 (Vertex permutation). Any affine bijective mapping $\widehat{T}: \hat{S}^{d-1} \rightarrow \hat{S}^{d-1}$ is associated with the permutation $\sigma$ of the set $\{0:d-1\}$ s.t. $\widehat{T}(\xi_i) := \varepsilon_{\sigma(i)}$ for all $i \in \{0:d-1\}$, where $\{\xi_i\}_{i \in \{0:d-1\}}$ are the vertices of $\hat{S}^{d-1}$. Then the above invariance holds true holds true iff all the vertices of $\hat{S}^{d-1}$ play symmetric roles when defining the basis functions $\{\zeta_m\}_{m \in \{1:n^{sh}_k\}}$.

For instance, for $d = 2$, $\hat{S}^1 = [0,1]$, and $k = 1$, the basis $\{1,s\} \in \mathbb{P}_{1,1}$ is not invariant w.r.t. vertex permutation, but the basis $\{1-s,s\}$ is. \square
A graphic representation of the dofs is shown in Figure 14.3. The number of arrows on a face counts the number of moments of the normal component considered over the face. The number of pairs of grey circles inside the triangle counts the number of moments inside the cell (one circle for the component along $\nu_{K,1}$ and one along $\nu_{K,2}$).

**Fig. 14.3** Degrees of freedom of $\mathbf{RT}_{k,d}$ finite elements for $d = 2$ and $k = 1$ (left) or $k = 2$ (right) (assuming that all the normals point outward).

**Lemma 14.14 (Face unisolvence).** Let $v \in \mathbf{RT}_{k,d}$. For all $F \in \mathcal{F}_K$, we have

$$\left[ \sigma(v) = 0, \forall \sigma \in \Sigma_F \right] \iff \left[ v_F \cdot \nu_F = 0 \right]. \quad (14.7)$$

**Proof.** The condition $\sigma(v) = 0$ for all $\sigma \in \Sigma_F$ means that $v_F \cdot \nu_F$ is orthogonal to $P_{k,d-1} \circ \mathbf{T}_F^{-1}$. Owing to Lemma 14.7 this is equivalent to $v_F \cdot \nu_F = 0$. \hfill $\square$

**Proposition 14.15 (Finite element).** $(K, \mathbf{RT}_{k,d}, \Sigma)$ is a finite element.

**Proof.** We have already shown the claim for $k = 0$, so we now consider $k \geq 1$. Observe first that the cardinality of $\Sigma$ can be evaluated as follows:

$$\text{card}(\Sigma) = d n_{sh} + (d + 1) n_{sh} = d \binom{d + k - 1}{k - 1} + (d + 1) \binom{d + k - 1}{k}$$

$$= \frac{(d + k - 1)!}{(d - 1)!(k - 1)!} \left( 1 + \frac{d + 1}{k} \right) = \dim(\mathbf{RT}_{k,d}).$$

Hence the statement will be proved once it is established that zero is the only function in $\mathbf{RT}_{k,d}$ that annihilates the dofs in $\Sigma$. Let $v \in \mathbf{RT}_{k,d}$ be such that $\sigma(v) = 0$ for all $\sigma \in \Sigma$. Owing to Lemma 14.14, we infer that $v_F \cdot \nu_F = 0$ for all $F \in \mathcal{F}_K$. This in turn implies that $\int_K v \cdot (\nabla \nabla \cdot v) \, dx = - \int_K (\nabla \cdot v)^2 \, dx$. Observing that $\nabla \nabla \cdot v$ is in $\mathbf{P}_{k-1,d}$ (recall that $\nabla \cdot v \in \mathbf{P}_{k,d}$ from Lemma 14.9), the assumption that $\sigma(v) = 0$ for all $\sigma \in \Sigma^c$ (i.e., $v$ is orthogonal to $\mathbf{P}_{k-1,d}$), together with the above identity imply that $\nabla \cdot v = 0$. Using Lemma 14.9, we conclude that $v \in \mathbf{P}_{k,d}$ and $v_F \cdot \nu_F = 0$ for all $F \in \mathcal{F}_K$.

Let $j \in \{1:d\}$. Since $\nu_{K,j} = \nu_{F_j} = |F_j| n_{F_j}$ for some face $F_j \in \mathcal{F}_K$, we infer that $\nu(x) \cdot \nu_{K,j} = \lambda_j(x) r_j(x)$ for all $x \in K$, where $\lambda_j$ is the barycentric coordinate of $K$ associated with the vertex opposite to $F_j$ (i.e., $\lambda_j$ vanishes
on $F_j$) and $r_j \in \mathbb{R}_{k-1,d}$; see Exercise 7.3(iv). The condition $\sigma(v) = 0$ for all $\sigma \in \Sigma^e$ implies that $\int_K (v \cdot \nu_{K,j}) r_j \, dx = 0$, which in turn means that $0 = \int_K (v \cdot \nu_{K,j}) r_j \, dx = \int_K \lambda_j r_j^2 \, dx$, thereby proving that $r_j = 0$ since $\lambda_j$ is positive in the interior of $K$. Hence $v \cdot \nu_{K,j}$ vanishes identically for all $j \in \{1:d\}$. This proves that $v = 0$ since $\{\nu_{K,j}\}_{j \in \{1:d\}}$ is a basis of $\mathbb{R}^d$. □

The shape functions $\{\theta_i\}_{i \in \mathcal{N}}$ associated with the dofs $\{\sigma_i\}_{i \in \mathcal{N}}$ defined in (14.5) can be constructed by choosing a basis $\{\phi_i\}_{i \in \mathcal{N}}$ of the polynomial space $\mathcal{RT}_{k,d}$ and by inverting the corresponding generalized Vandermonde matrix $\mathcal{V}$ as explained in Proposition 5.5. Recall that this matrix has entries $V_{ij} = \sigma_j(\phi_i)$ and that the $i$-th line of $\mathcal{V}^{-1}$ gives the components of the shape function $\theta_i$ in the basis $\{\phi_i\}_{i \in \mathcal{N}}$. The basis $\{\phi_i\}_{i \in \mathcal{N}}$ chosen in Bonazzoli and Rapetti [31] (built by dividing the simplex into smaller sub-simplices following the ideas in Rapetti and Bossavit [155], Christiansen and Rapetti [65]) is particularly interesting since the entries of $\mathcal{V}^{-1}$ are integers. One could also choose $\{\phi_i\}_{i \in \mathcal{N}}$ to be the hierarchical basis of $\mathcal{RT}_{k,d}$ constructed in Fuentes et al. [99, §7.3]. This basis can be organized into functions attached to the faces of $K$ and to $K$ itself in such a way that the generalized Vandermonde matrix $\mathcal{V}$ is block-triangular (notice though that this matrix is not block-diagonal).

**Remark 14.16 (Dof independence).** As in Remark 7.20, we infer from Exercise 5.2 that the interpolation operator $\mathcal{I}_h^d$ associated with the $\mathcal{RT}_{k,d}$ element is independent of the bases $\{\zeta_m\}_{m \in \{1:n_h^0\}}$ and $\{\psi_m\}_{m \in \{1:n_h^1\}}$ used to define the dofs in (14.5). This operator is also independent of the mappings $\mathcal{T}_F$ and of the orientation vectors $\{\nu_F\}_{F \in \mathcal{F}_K}$ and $\{\nu_{K,j}\}_{j \in \{1:d\}}$. □

**Remark 14.17 (Literature).** The $\mathcal{RT}_{k,d}$ finite element has been introduced in Raviart and Thomas [156, 157] for $d = 2$; see also Weil [189, p. 127], Whitney [190, Eq. (12), p. 139] for $k = 0$. The generalization to $d \geq 3$ is due to Nédélec [143]. The reading of [143] is highly recommended; see also Boffi et al. [29, §2.3.1], Hiptmair [113], Monk [138, pp. 118-126]. The name Raviart–Thomas seems to be an accepted practice in the literature. □

### 14.4 Generation of Raviart–Thomas elements

Let $\hat{K}$ be the reference simplex in $\mathbb{R}^d$. Let $\mathcal{T}_h$ be an affine simplicial mesh. Let $K = \mathcal{T}_K(\hat{K})$ be a mesh cell where $\mathcal{T}_K : \hat{K} \to K$ is the geometric mapping and let $J_K$ be the Jacobian matrix of $\mathcal{T}_K$. Let $F \in \mathcal{F}_K$ be a face of $K$. We have $F = \mathcal{T}_K(\hat{F})$ for some face $\hat{F} \in \mathcal{F}_{\hat{K}}$. Owing to Theorem 10.8, it is possible (using the increasing vertex-index enumeration) to orient the faces $F$ and $\hat{F}$ in a way that is compatible with the geometric mapping $\mathcal{T}_K$. This means that the unit normal vectors $n_F$ and $\hat{n}_{\hat{F}}$ satisfy (10.6b), i.e., $n_F = \Phi_K^d(\hat{n}_{\hat{F}})$ with $\Phi_K^d$ defined in (9.14a). In other words we have
Lemma 14.18 (Transformation of dofs).

Let $C$ we have $F$ respectively, sharing the vertex with the lowest enumeration index, so that

$$n_F \circ T_{K|\hat{F}} = \frac{1}{\|J_K^n\|_F^2} J_K^{-T} \hat{n}_{\hat{F}},$$

where $\epsilon_K := \frac{\det(J_K)}{\|\det(J_K)\|} = \pm 1$. Recalling that $\nu_F := |F| n_F$, $\nu_{\hat{F}} := |\hat{F}| \hat{n}_{\hat{F}}$ and that $|F| = \|\det(J_K)\| \|J_K^{-T} \hat{n}_{\hat{F}}\|_F |\hat{F}|$ owing to Lemma 9.12, we infer that

$$\nu_F \circ T_{K|\hat{F}} = \det(J_K) J_K^{-T} \nu_{\hat{F}}. \quad (14.9)$$

Due to the role played by the normal component at the faces, we consider the contravariant Piola transformation for vector-valued fields s.t.

$$\psi^d_K(v) := \det(J_K)^{-1} (v \circ T_K), \quad (14.10)$$

and the pullback by the geometric mapping for scalar fields s.t. $\psi^K(q) := q \circ T_K$. Finally, we orient $K$ and $\hat{K}$ by the $d$ vectors $\{\nu_{K,j} := |F_j| n_{F_j}\}_{j \in \{1:d\}}$ and $\{\nu_{\hat{K},j} := |\hat{F}_j| \hat{n}_{\hat{F}_j}\}_{j \in \{1:d\}}$ corresponding to the $d$ faces of $K$ and $\hat{K}$, respectively, sharing the vertex with the lowest enumeration index, so that we have $F_j = T_K(\hat{F}_j)$ for all $j \in \{1:d\}$. The above considerations show that $\nu_{K,j} \circ T_K = \det(J_K) J_K^{-T} \nu_{\hat{K},j}$.

**Lemma 14.18 (Transformation of dofs).** Let $v \in C^0(K)$ and let $q \in C^0(K)$. The following holds true:

\[
\frac{1}{|F|} \int_F (v \cdot \nu_F) q \, ds = \frac{1}{|F|} \int_{\hat{F}} (\psi^d_K(v) \cdot \nu_{\hat{F}}) \psi^K_q(q) \, d\hat{s}, \quad \forall F \in \mathcal{F}_K, \quad (14.11a)
\]

\[
\frac{1}{|K|} \int_K (v \cdot \nu_{K,j}) q \, dx \quad = \quad \frac{1}{|K|} \int_{\hat{K}} (\psi^d_K(v) \cdot \nu_{\hat{K},j}) \psi^K_q(q) \, d\hat{x}, \quad \forall j \in \{1:d\}. \quad (14.11b)
\]

**Proof.** The identity (14.11a) is nothing but (10.7a) from Lemma 10.4, which itself is a reformulation of (9.15a) from Lemma 9.13 (the fact that $T_K$ is affine is not used here). The proof of (14.11b) is similar since

$$\int_K (v \cdot \nu_{K,j}) q \, dx = \int_{\hat{K}} (v \circ T_K) \cdot (\nu_{K,j} \circ T_K) \psi^K_q(q) |\det(J_K)| d\hat{x}$$

$$= \int_{\hat{K}} (\psi^d_K(v) \cdot \nu_{\hat{K},j}) \psi^K_q(q) |\det(J_K)| d\hat{x},$$

and since $T_K$ is affine, we have $|K| = |\det(J_K)| |\hat{K}|$. \qed

**Proposition 14.19 (Generation).** Let $(\hat{K}, \hat{P}, \hat{\Sigma})$ be a simplicial $RT_{k,d}$ element with face and cell dofs defined using the polynomial bases $\{\zeta_m\}_{m \in \{1:n^d_{\hat{m}}\}}$ and $\{\psi_m\}_{m \in \{1:n^c_{\hat{m}}\}}$ (if $k \geq 1$) of $P_{k,d-1}$ and $P_{k-1,d}$, respectively; see (14.5).

Assume that the geometric mapping $T_K$ is affine and that (14.9) holds true. Then the finite element $(K, P_K, \Sigma_K)$ generated using Proposition 9.2 with the transformation (14.10) is a simplicial $RT_{k,d}$ finite element with dofs.
where $T_{K,F} := T_{K|F} \circ T_{F}$ is the affine bijective mapping from $\hat{S}^{d-1}$ to $F$ that maps the $d$ vertices of $\hat{S}^{d-1}$ to the $d$ vertices of $F$ with increasing indices.

Proof. See Exercise 14.4 for the proof that $P_K = RTRTRT$.

Remark 14.20 (Unit). Since the unit of all the dofs is $L^{d-1}$, the shape functions scale as $L^{1-d}$ where $L$ is a length unit.

Remark 14.21 (Non-affine meshes). Proposition 9.2 together with the map (14.10) can still be used to generate a finite element $(K, P_K, \Sigma_K)$ if the geometric mapping $T_K$ is non-affine. The function space $P_K$ and the dofs in $\Sigma_K$ then differ from those of the $RTRTRT$ element.

14.5 Other $H(\text{div})$ finite elements

14.5.1 Brezzi–Douglas–Marini elements

Brezzi–Douglas–Marini (BDM) elements [47, 48] offer an interesting alternative to Raviart–Thomas elements since they allow one to work with finite elements $(K, P, \Sigma)$ where the polynomial space is $P := P_{k,d} \subset RTRTRT_{k,d}$, $k \geq 1$, is optimal from the approximation viewpoint. The price to pay is that the divergence operator $\nabla \cdot$ is surjective from $P_{k,d-1}$ onto $P_{k-1,d}$ only. This is not a limitation if the functions to be interpolated have a simple divergence, e.g., if they are divergence-free.

Let $K$ be a simplex in $\mathbb{R}^d$. The dofs of BDM elements are attached to the $(d+1)$ faces of $K$ and to $K$ itself (for $k \geq 2$). The face dofs are the same as for Raviart–Thomas elements, i.e., the linear forms $\sigma_{F,m}^i(v)$ defined in (14.5a) for all $F \in \mathcal{F}_K$ and every $m \in \{1:n_{sh}^i\}$ with $n_{sh}^i := \dim(P_{k,d-1})$. Note that the cell dofs for Raviart–Thomas elements are moments against a set of basis functions of $P_{k-1,d}$, whereas those for BDM elements are moments against a set of basis functions of the Nédélec polynomial space $N_{k-2,d}$ introduced in the next chapter (see §15.2). At this stage it is sufficient to know that $P_{k-2,d} \subseteq N_{k-2,d} \subseteq P_{k-1,d}$ and that $\dim(N_{k-2,2}) = (k-1)(k+1)$ and $\dim(N_{k-2,3}) = \frac{1}{2}(k-1)(k+1)(k+2)$ (see Lemma 15.7). We define

\[
\tilde{\sigma}_{m}^c(v) := \int_{K} v \cdot \tilde{\psi}_m \, dx, \quad \forall m \in \{1:n_{sh}^c\},
\]
where \( \{ \tilde{\psi}_m \}_{m \in \{1: \tilde{n}_{sh} \}} \) is a basis of \( \mathbf{N}_{k-2,d} \) and \( \tilde{n}_{sh} := \dim(\mathbf{N}_{k-2,d}) \). Let us set \( \Sigma := \{ \sigma_{F,m}^1 \}_{F \in \mathcal{F}_K, m \in \{1:n_{sh}\}} \cup \{ \tilde{\sigma}^c_m \}_{m \in \{1: \tilde{n}_{sh} \}} \).

**Proposition 14.22 (Finite element).** \((K, \mathcal{P}_{k,d}, \Sigma)\) is a finite element.

**Proof.** See Boffi et al. [29, p. 88]. \(\square\)

Hierarchical basis functions for the BDM element are constructed in Ainsworth and Coyle [6], Schöberl and Zaglmayr [168].

**Remark 14.23 (Generation).** Generating BDM elements also involves the covariant Piola transformation \( \psi_K(w) := \int_T^K(w \circ T_K) \) defined in (9.9b), so that \( \int_K \nabla \tilde{\psi}_m \, dx = \epsilon_K \int_K \psi^1_K(w) \psi^c_K(\tilde{\psi}_m) \, dx \) with \( \epsilon_K := \frac{\det(\hat{\eta}(K))}{\det(\hat{\eta}(\mathcal{P}))} = \pm 1. \) \(\square\)

### 14.5.2 Cartesian Raviart–Thomas elements

Let us briefly review the Cartesian Raviart–Thomas finite elements. We refer the reader to Exercise 14.6 for the proofs. For a multi-index \( \alpha \in \mathbb{N}^d \), we define the (anisotropic) polynomial space \( \mathcal{Q}_{\alpha_1,...,\alpha_d} \) composed of \( d \)-variate polynomials whose degree with respect to \( x_i \) is at most \( \alpha_i \) for all \( i \in \{1:d\} \). Let \( k \in \mathbb{N} \) and define

\[
\mathbf{RT}^{\square}_{k,d} := \mathcal{Q}_{k+1,k,...,k} \times \cdots \times \mathcal{Q}_{k,...,k,k+1}.
\]  

(14.14)

One can verify that \( \dim(\mathbf{RT}^{\square}_{k,d}) = d(k + 2)(k + 1)^{d-1} \) and that

\[
\nabla \cdot \mathbf{v} \in \mathcal{Q}_{k,d}, \quad \mathbf{v}_{\mathcal{H}} \in \mathcal{Q}_{k,d-1} \circ \mathbf{T}_{\mathcal{H}}^{-1},
\]  

(14.15)

for all \( \mathbf{v} \in \mathbf{RT}^{\square}_{k,d} \) and every affine hyperplane \( \mathcal{H} \) in \( \mathbb{R}^d \) with normal vector \( \mathbf{v}_{\mathcal{H}} \) parallel to one of the vectors of the canonical basis of \( \mathbb{R}^d \) and where \( \mathbf{T}_{\mathcal{H}} : \mathbb{R}^{d-1} \to \mathcal{H} \) is an affine bijective mapping.

Let \( K \) be a cuboid in \( \mathbb{R}^d \). Each face \( F \in \mathcal{F}_K \) of \( K \) is oriented by the normal vector \( \mathbf{v}_F \) with \( \|\mathbf{v}_F\|_2 = |F| \). Let \( \mathbf{T}_F \) be an affine bijective mapping from \( [0,1]^{d-1} \) onto \( F \). Let us orient \( K \) using \( \mathbf{v}_{K,j} := |F_j| \mathbf{e}_j \) for all \( j \in \{1:d\} \), where \( \{\mathbf{e}_j\}_{j \in \{1:d\}} \) is the canonical basis of \( \mathbb{R}^d \) and \( |F_j| \) is the measure of any of the two faces of \( K \) supported in a hyperplane perpendicular to \( \mathbf{e}_j \). Let \( \Sigma \) be the set composed of the following linear forms:

\[
\sigma^1_{F,m}(\mathbf{v}) := \frac{1}{|F|} \int_F (\mathbf{v} \cdot \mathbf{v}_F)(\zeta_m \circ \mathbf{T}_F^{-1}) \, ds, \quad \forall F \in \mathcal{F}_K,
\]  

(14.16a)

\[
\sigma^c_{j,m}(\mathbf{v}) := \frac{1}{|K|} \int_K (\mathbf{v} \cdot \mathbf{v}_{K,j}) \psi_{j,m} \, dx, \quad \forall j \in \{1:d\},
\]  

(14.16b)

where \( \{\zeta_m \}_{m \in \{1:n^f_{sh}\}} \) is a basis of \( \mathcal{Q}_{k,d-1} \) with \( n^f_{sh} := (k + 1)^{d-1} \), and \( \{\psi_{j,m} \}_{m \in \{1:n^c_{sh}\}} \) is a basis of \( \mathcal{Q}_{k,...,k,k-1,...,k} \) with \( n^c_{sh} := k(k + 1)^{d-1} \) if \( k \geq 1 \), with the index \( (k - 1) \) at the \( j \)-th position for all \( j \in \{1:d\} \).
Proposition 14.24 (Finite element). \((K, \mathbf{RT}_0^{d}, \Sigma)\) is a finite element.

Cartesian Raviart–Thomas elements can be generated for all the mesh cells of an affine mesh composed of parallelepipeds by using affine geometric mappings and the contravariant Piola transformation (recall however that orienting such meshes requires some care; see Theorem 10.10).

Example 14.25 (Shape functions and dofs for \(\mathbf{RT}_0^{d}\)). Let \(K := [0, 1]^d\). Let \(F_i\) and \(F_{d+i}\) be the faces defined by \(x_i = 0\) and \(x_i = 1\), respectively, for all \(i \in \{1:d\}\). Using the basis function \(\zeta_1 := 1\) for \(Q_0^{d−1}\), the \(2d\) dofs are the mean-value of the normal component over each face of \(K\), and the shape functions are \(\theta_{F_i}(x) := (1−x_i)n_{F_i}\) and \(\theta_{d+i}(x) := x_in_{F_i}\) for all \(i \in \{1:d\}\).

The dofs are illustrated in Figure 14.4.

\(\Box\)

Remark 14.26 (Other elements). Alternative elements are the Cartesian Brezzi–Douglas–Marini elements in dimension two, the Brezzi–Douglas–Durán–Fortin ones in dimension three (see [47, 48]) and their reduced versions by Brezzi–Douglas–Fortin–Marini [49].

Exercises

Exercise 14.1 (\(\mathbf{RT}_0^{d}\)). (i) Prove that \(\int_K \theta_F^d \, dx = c_F−c_K\), where \(c_F\) and \(c_K\) are the barycenters of \(f\) and \(K\), respectively. (Hint: use (14.3) and \(\int_F x \, ds = |F|c_F\).) Provide a second proof without using (14.3). (Hint: Fix \(e \in \mathbb{R}^d\), define \(\phi(x) := (x−c_F)\cdot e\), observe that \(\nabla \phi = e\), and compute \(e \cdot \int_K \theta_F^d \, dx\).) (ii) Prove that \(\sum_{F \in \mathcal{F}_K} |F| \theta_F^d \otimes n_{K|F} = \mathbb{I}_d\) for all \(x \in K\). (Hint: use (7.1).) (iii) Prove that \(v(x) = \langle v \rangle_K + \frac{1}{2} \nabla v(x−c_K)\) for all \(v \in \mathbf{RT}_0^{d}\), where \(\langle v \rangle_K := \frac{1}{|K|} \int_K v \, dx\) is the mean-value of \(v\) on \(K\).

Exercise 14.2 (\(\mathbf{RT}_0^{d}\) in 3D). Let \(d = 3\). Let \(F_i, \ i \in \{0:3\}\), be a face of \(K\) with vertices \(\{a_r, a_p, a_q\}\) s.t. \((z_q−z_r) \times (z_p−z_r)\cdot n_{K|F_i} > 0\). (i) Prove that
\( \nabla \lambda_p \times \nabla \lambda_q = \frac{2}{|K|} \) and prove similar formulas for \( \nabla \lambda_q \times \nabla \lambda_r \) and \( \nabla \lambda_r \times \nabla \lambda_p \).

(Hint: Prove the formula in the reference simplex, then use Exercise 9.6.)

(ii) Prove that \( \theta_i^d = -2(\lambda_p \nabla \lambda_q \times \nabla \lambda_r + \lambda_q \nabla \lambda_r \times \nabla \lambda_p + \lambda_r \nabla \lambda_p \times \nabla \lambda_q) \).

Find the counterpart of this formula for \( d = 2 \).

Exercise 14.3 (Piola transformation). (i) Let \( v \in C^1(K) \) and \( q \in C^0(K) \). Prove that
\[
\int_K q \nabla \cdot v \, dx = \int_{\mathring{K}} \psi_K^q \nabla \cdot \psi_K^v \, d\mathring{x}.
\]
(ii) Show that
\[
\int_K v \cdot \theta \, dx = \epsilon_K \int_{\mathring{K}} \psi_K^v \cdot \psi_K^\theta \, d\mathring{x}
\]
for all \( \theta \in C^1(K) \).

Exercise 14.4 (Generating \( \mathbf{RT}_{k,d} \)). (i) Let \( c \in \mathbb{R}^d \), \( q \in P^H_{k,d} \), and \( \Lambda \in \mathbb{R}^{d \times d'} \). Show that there is \( r \in \mathbb{R}^{k-1,d'} \) such that \( q(\Lambda y + c) = q(\Lambda y) + r(y) \). (ii) Defining \( s(y) := q(\Lambda y) \), show that \( s \in P^H_{k,d} \). (iii) Prove that \( (\psi_K^s)^{-1}(\mathbf{RT}_{k,d}) \subset \mathbf{RT}_{k,d} \). (iv) Prove the converse inclusion.

Exercise 14.5 (BDM). Verify that \( \text{card}(\Sigma) = \dim(\mathbb{P}_{k,d}) \) for \( d \in \{2, 3\} \).

Exercise 14.6 (Cartesian Raviart–Thomas element). (i) Propose a basis for \( \mathbf{RT}_{0,2} \) and for \( \mathbf{RT}_{0,3} \) in \( K := [0,1]^d \), and show that \( \dim(\mathbf{RT}_{k,d}) = d(k+2)(k+1)^{d-1} \). (ii) Prove (14.15). (iii) Prove Proposition 14.24.
Solution to exercises

Exercise 14.1 (\(\mathbf{RT}_{0,d}\)). (i) By definition we have \(\int_K \boldsymbol{\theta}_F^i \cdot dx = \frac{1}{|K|}(c_K - z_F)\) since \(\int_K x \cdot dx = |K|c_K\). Let us prove that \(c_K - z_F = d(c_F - c_K)\). Since \(c_K = \frac{1}{d+1} \sum_{F \in \mathcal{F}_K} z_F\), we infer that

\[
d(c_F - c_K) = \left(\sum_{F' \in \mathcal{F}_K \setminus F} z_{F'}\right) - dc_K = \left(\sum_{F' \in \mathcal{F}_K} z_{F'}\right) - z_F - dc_K = (d+1)c_K - z_F - dc_K = c_K - z_F.
\]

Hence we have

\[
\int_K \boldsymbol{\theta}_F^i \cdot dx = \frac{1}{d}(c_K - z_F) = c_F - c_K.
\]

For the second proof, let \(e \in \mathbb{R}^d\). Let \(\phi(x) = (x - c_F) \cdot e\) and observe that \(\nabla \phi = e\). Then,

\[
e \cdot \int_K \boldsymbol{\theta}_F^i \cdot dx = \int_K \boldsymbol{\theta}_F^i \cdot \nabla \phi \cdot dx = -\int_K \phi \nabla \cdot \boldsymbol{\theta}_F^i \cdot dx + \sum_{F' \in \mathcal{F}_K} \int_{F'} (\theta_F^i \cdot n_{K|F'}) \phi \cdot ds.
\]

Owing to Lemma 14.7, \(\theta_F^i \cdot n_{K|F'}\) is piecewise constant and equal to \(\theta_F^i \cdot n_{K|F'} = \frac{\delta_{F,F'}}{|F|}\). Moreover we have \(|K| \nabla \cdot \theta_F^i = \int_K \nabla \cdot \theta_F^i \cdot dx = \int_K \theta_F^i \cdot n_{K|F} \cdot ds = 1\). We infer that

\[
e \cdot \int_K \boldsymbol{\theta}_F^i \cdot dx = \frac{1}{|K|} \int_K \phi \cdot dx + \frac{1}{|F|} \int_F \phi \cdot ds = -(c_K - c_F) \cdot e,
\]

since \(\int_K \phi \cdot dx = \phi(c_K)|K|\) and \(\int_F \phi \cdot ds = 0\). This implies that \(\int_K \theta_F^i \cdot dx = c_F - c_K\) since the above equality holds true for all \(e \in \mathbb{R}^d\).

(ii) Let \(x \in K\). We observe that

\[
\sum_{F \in \mathcal{F}_K} |F| \theta_F^i(x) \otimes n_{K|F} = \sum_{F \in \mathcal{F}_K} \frac{|F|}{d|K|} (x - z_F) \otimes n_{K|F}
\]

\[
= \sum_{F \in \mathcal{F}_K} \frac{|F|}{d|K|} (c_K - z_i) \otimes n_{K|F}
\]

\[
= \sum_{i \in \mathcal{F}_K} \frac{|F|}{|K|} (c_F - c_K) \otimes n_{K|F} = I_d,
\]

where we used the definition of \(\theta_F^i\), the first geometric identity in (7.1) to replace \(x\) by \(c_K\), the fact that \(c_K - z_F = d(c_F - c_K)\), and the second geometric identity in (7.1) to conclude.

(iii) Let \(v \in \mathbf{RT}_{0,d}\). Then we can write \(v = a + b(x - c_K)\), where \(a \in \mathbb{R}^d\), \(b \in \mathbb{R}\), whence we infer that \(\nabla v = bd\), i.e., \(b = \frac{1}{d} \nabla v\). Moreover, since
(x - c_K) has zero mean-value on K, we infer that a = (v)_K. In conclusion v = (v)_K + \frac{1}{2}(\nabla \cdot v)(x - c_K).

Exercise 14.2 (RT_0,d in 3D). (i) Let us note first that the assumption that (z_q - z_r) × (z_p - z_r) × n_K|_K > 0 means that the vectors (z_p - z_r), (z_q - z_r), (z_i - z_r) form a right-handed triple. Let us do the computation in the reference simplex. Let \( z_r = 0, \hat{z}_p - \hat{z}_r = (1, 0, 0)^T, \hat{z}_q - \hat{z}_r = (0, 1, 0)^T, \) and \( \hat{z}_i - \hat{z}_r = (0, 0, 1). \) Then \( \lambda_p = x_1, \lambda_q = x_2, \nabla \hat{\lambda}_p = (1, 0, 0)^T \) and \( \hat{\lambda}_q = (0, 1, 0)^T. \) This implies that \( \nabla \hat{\lambda}_p \times \nabla \hat{\lambda}_q = (0, 0, 1) = \hat{z}_i - \hat{z}_r. \) But \( 6|K| = 1. \) Hence \( \nabla \hat{\lambda}_p \times \nabla \hat{\lambda}_q = \frac{\hat{z}_i - \hat{z}_r}{6|K|}. \) Let us now prove the formula in K. Let \( T_K \) be the affine mapping that transforms \( (\hat{z}_p, \hat{z}_q, \hat{z}_r, \hat{z}_i) \) into \( (z_p, z_q, z_r, z_i), \) respectively. Let \( J_K \) be the Jacobian matrix of \( T_K. \) Observe that \( \det(J_K) > 0 \) since \( (\hat{z}_p - \hat{z}_r), (\hat{z}_q - \hat{z}_r), (z_p - z_r), (z_q - z_r), (z_i - z_r) \) form two right-handed triples. From Exercise 9.6, we infer that

\[
\nabla \hat{\lambda}_p \times \nabla \hat{\lambda}_q = (J_K^{-1})^T \nabla \hat{\lambda}_p \times (J_K^{-1})^T \nabla \hat{\lambda}_q = \det(J_K^{-1}) J_K \nabla \hat{\lambda}_p \times \nabla \hat{\lambda}_q = \det(J_K^{-1}) \frac{\hat{z}_i - \hat{z}_r}{6|K|},
\]

which proves that \( \nabla \hat{\lambda}_p \times \nabla \hat{\lambda}_q = \frac{\hat{z}_i - \hat{z}_r}{6|K|} \) since \( \det(J_K) = \frac{|J|}{|K|}. \) By circular permutation on the indices \((p, q, r)\) (which does not change the orientation of K), we also have \( \nabla \hat{\lambda}_q \times \nabla \hat{\lambda}_r = \frac{\hat{z}_i - \hat{z}_r}{6|K|} \) and \( \nabla \hat{\lambda}_r \times \nabla \hat{\lambda}_p = \frac{\hat{z}_i - \hat{z}_r}{6|K|} \).

(ii) Recall that \( \theta^d_i = \frac{\hat{z}_i - \hat{z}_r}{6|K|} \) and

\[
x - z_i = \lambda_p(x)(z_p - z_i) + \lambda_q(x)(z_q - z_i) + \lambda_r(x)(z_r - z_i).
\]

It follows immediately from (i) that

\[
\theta^d_i(x) = -2(\lambda_p(x)\nabla \hat{\lambda}_q \times \nabla \hat{\lambda}_r + \lambda_q(x)\nabla \hat{\lambda}_r \times \nabla \hat{\lambda}_p + \lambda_r(x)\nabla \hat{\lambda}_p \times \nabla \hat{\lambda}_q).
\]

Exercise 14.3 (Piola transformation). (i) This follows (9.8c), i.e., \( \nabla \cdot v(x) = \frac{1}{\det(J_K(x))} \nabla \cdot \psi_K(v)(\bar{x}). \) (ii) We prove the second identity as follows

\[
\int_K v \cdot \theta \, dx = \int_{\bar{K}} (v \circ T_K) \cdot (\theta \circ T_K) |\det J_K| \, d\bar{x} = \epsilon_K \int_{\bar{K}} (J_K^{-1} v \circ T_K) \cdot (J_K^T \theta \circ T_K) \, d\bar{x} = \epsilon_K \int_{\bar{K}} \psi^d_K(v) \cdot \psi_K^e(\theta) \, d\bar{x}.
\]

Exercise 14.4 (Generating RT_{k,d}). (i) Let \( x, c \in \mathbb{R}^d \) and consider the polynomial \( q(x) = \sum |\alpha| = d a_\alpha x_1^{\alpha_1} \ldots x_d^{\alpha_d}. \) We have
\[ q(x + c) = \sum_{|\alpha| = d} a_\alpha (x_1 + c_1)^{\alpha_1} \ldots (x_d + c_d)^{\alpha_d} \]

\[ = \sum_{|\alpha| = d} a_\alpha (x_1^{\alpha_1} + r_1(x_1)) \ldots (x_d^{\alpha_d} + r_d(x_d)) \]

where \( r_i \in \mathbb{P}_{\alpha_i - 1,d}, i \in \{1:d\} \). We infer that

\[ q(x + c) = \sum_{|\alpha| = d} a_\alpha x_1^{\alpha_1} \ldots x_d^{\alpha_d} + t(x) = q(x) + t(x), \]

where \( t \in \mathbb{P}_{k-1,d} \). Replacing \( x \) by \( \lambda y \) we obtain

\[ q(\lambda y + c) = q(\lambda y) + t(\lambda y). \]

But defining \( r \) such that \( r(y) = t(\lambda y) \), we have \( r \in \mathbb{P}_{k-1,d} \).

(ii) Let

\[ p_i(y) = \left( \sum_{j \in \{1:d\} \setminus k,d} \lambda_{ij} y_j \right)^{\alpha_i}. \]

This polynomial is homogeneous of degree \( \alpha_i \). Moreover the product of a homogeneous polynomial of degree \( \alpha_i \) with a homogeneous polynomial of degree \( \alpha_j \) is a homogeneous polynomial of degree \( \alpha_i + \alpha_j \). Hence

\[ q(\lambda y) = \sum_{|\alpha| = d} a_\alpha p_1(y) \ldots p_d(y) \]

is homogeneous of degree \( \alpha_1 + \ldots + \alpha_d = |\alpha| = k \).

(iii) Let \( T_K(\bar{x}) = J_K(\bar{x} + b_K) \) with \( J_K \in \mathbb{R}^{d \times d} \) and \( b_K \in \mathbb{R}^d \). Let \( v \) be a member of \((\psi_K^d)^{-1}(RT_{k,d})\). Then \( \psi_K^d(v) = \bar{p} + \bar{\alpha} \bar{q} \) with \( \bar{p} \in \mathbb{P}_{k,d} \) and \( \bar{q} \in \mathbb{P}_{k,d}^H \), yielding

\[ v = (\psi_K^d)^{-1}(\bar{p} + \bar{\alpha} \bar{q}) = \frac{1}{\det(J_K)} J_K(\bar{p} \circ T_K^{-1} + (\bar{\alpha} \bar{q}) \circ T_K^{-1}). \]

Using \( \bar{x} = J_K^{-1}(x - b_K) \), we have \( \bar{q} \circ T_K^{-1} = \bar{q}(J_K^{-1} x - b_K) = \bar{q}(J_K^{-1} x) + r \) where \( r \in \mathbb{P}_{k-1,d} \), and we have shown that \( \bar{q} \circ J_K^{-1} \in \mathbb{P}_{k,d}^H \). Hence \( v = s + \frac{1}{\det(J_K)} \int_K J_K^{-1} x (\bar{q} \circ J_K^{-1}) = s + xt \), where \( s \in \mathbb{P}_{k,d} \) and \( t \in \mathbb{P}_{k,d}^H \). As a result, \((\psi_K^d)^{-1}(RT_{k,d}) \subset RT_{k,d}\).

(iv) The converse follows from a dimension argument.

**Exercise 14.5 (BDM).** For \( d = 2 \), we have \( \text{card}(\Sigma) = 3(k+1) + (k-1)(k+1) = (k+1)(k+2) = \dim(\mathbb{P}_{k,2}) \). For \( d = 3 \), we have \( \text{card}(\Sigma) = \frac{4}{2}(k+1)(k+2) + \frac{1}{2}(k-1)(k+1)(k+2) = \frac{1}{2}(k+1)(k+2)(k+3) = \dim(\mathbb{P}_{k,3}) \).
Exercise 14.6 (Cartesian Raviart–Thomas element). (i) A basis for $\mathbf{RT}_{0,2}^d$ is $(\frac{1}{0}, \frac{0}{1}, \frac{x_1}{0}, \frac{0}{x_2})$, while a basis for $\mathbf{RT}_{0,3}^d$ is $(\frac{1}{0}, \frac{0}{1}, \frac{x_1}{0}, \frac{0}{x_2}, \frac{x_1^2}{0}, \frac{0}{x_2^2})$. Since $Q_{k+1,k,...,k}$ is a tensor product space, the dimension of $Q_{k+1,k,...,k}$ is equal to $(k+2)(k+1)^{d-1}$. This immediately implies that $\dim(\mathbf{RT}_{k,d}^3) = d(k+2)(k+1)^{d-1}$.

(ii) Let $v_1 \in Q_{k+1,k,...,k}$. Then $v_1(x) = \sum_{\alpha \in A_{1,k,d}} a_{\alpha} x_1^{\alpha_1} \ldots x_d^{\alpha_d}$, where $A_{1,k,d} := \{ (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d \mid \alpha_1 \leq k+1, \alpha_2, \ldots, \alpha_d \leq k \}$. Hence $\partial_1 v_1(x) = \sum_{\alpha} a_{\alpha} \alpha_1 x_1^{\alpha_1-1} \ldots x_d^{\alpha_d} \in Q_{k,d}$. The same reasoning on the other indices implies that $\nabla (\mathbf{RT}_{k,d}^3) \subset Q_{k,d}$. Let us prove $v_{H}^T n_{H} \in Q_{k,d-1} \circ T_{H}^{-1}$ for all $v \in \mathbf{RT}_{k,d}^3$. We do the proof for $n_{H} = e_1$, which means that $x_1$ is constant over $H$. Hence $v_{H}^T n_{H} = v_{1|H} = \sum_{\alpha \in A_{1,k,d}} (a_{\alpha} x_1^{\alpha_1}) x_2^{\alpha_2} \ldots x_d^{\alpha_d} = \sum_{\beta \in B_{k,d}} b_{\beta} x_2^{\beta_1} \ldots x_d^{\beta_d}$, where $B_{k,d} := \{ (\beta_1, \ldots, \beta_{d-1}) \in \mathbb{N}^{d-1} \mid \beta_1, \ldots, \beta_{d-1} \leq k \}$. Let $T_{H} : \mathbb{R}^{d-1} \rightarrow H$ be defined by $T_{H}(y_1, \ldots, y_{d-1}) := (x_1, y_1, \ldots, y_{d-1})$. Then $T_{H}^{-1}(x) = (x_2, \ldots, x_d)$. Let $q(y) := \sum_{\beta \in B_{k,d}} b_{\beta} y_1^{\beta_1} \ldots y_{d-1}^{\beta_d}$. Then $v_{H}^T n_{H} = q \circ T_{H}^{-1}$ where $q \in Q_{k,d-1}$.

(iii) Observe first that $\text{card}(\Sigma) = dk(k+1)^{d-1} + 2d(k+1)^{d-1} = d(k+1)^{d-1}(k+2) = \dim(\mathbf{RT}_{k}^d)$. Let $v \in \mathbf{RT}_{k}^d$ be such that $\sigma(v) = 0$ for all $\sigma \in \Sigma$. The assumption $\sigma_{i,m}^T(v) = 0$ for all $i \in \{1:2d\}$ and all $m \in \{1:n_{k}^{d}\}$, together with the fact that $v|_{F_i} n_{F_i} \in Q_{k,d-1} \circ T_{F_i}^{-1}$ implies that $v|_{F_i} n_{F_i} = 0$. This in turns implies that $v$ can be rewritten as follows: $v = (x_1 (1-x_1) r_1, \ldots, x_d (1-x_d) r_d)^T$ where $r := (r_1, \ldots, r_d)^T$ is a member of $Q_{k-1,k-1} \times \ldots \times Q_{k-1,k-1}$. Then the assumption $\sigma_{i,m}^T(v) = 0$ for all $i \in \{1:d\}$ and all $m \in \{1:n_{k}^{d}\}$ implies that $\int_K v \cdot r \, dx = 0$, which in turns leads to $r = 0$, thereby proving that $v = 0$. 