Quiz 1 (Notes, books, and calculators are not authorized)

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1: Let $\phi(x,y) = \int_0^{\sin(x)} \log(1+y^2+z^2)\mathrm{d}z$. Given $\alpha,\beta,\gamma \in \mathbb{R}$, compute $\partial_x\phi(\alpha,\alpha+\beta)$ and $\partial_y\phi(\alpha-\beta,\gamma)$. (Do not try to simplify the results).

Recalling the fundamental theorem of calculus

$$\boxed{\partial_t\left(\int_0^t f(z)\mathrm{d}z\right) = f(t),}$$

we apply the chain rule repeatedly

$$\partial_x\phi(x,y) = \log(1+y^2+\sin(x)^2)\cos(x)$$

This means that

$$\partial_x\phi(\alpha,\alpha+\beta) = \log(1+(\alpha+\beta)^2+\sin(\alpha)^2)\cos(\alpha)$$

Recalling that

$$\boxed{\partial_t\int_u^v f(s,t)\mathrm{d}s = \int_u^v \partial_t f(s,t)\mathrm{d}s,}$$

we apply the chain rule repeatedly

$$\partial_y\phi(x,y) = \int_0^{\sin(x)} \frac{2y}{1+y^2+z^2}\mathrm{d}z$$

This means that

$$\partial_y\phi(\alpha-\beta,\gamma) = \int_0^{\sin(\alpha-\beta)} \frac{2\gamma}{1+\gamma^2+z^2}\mathrm{d}z.$$

Question 2: Consider the heat equation $\partial_t T - k\partial_{xx}T = f(x)$, $x\in[a,b]$, $t>0$, with $f(x)=0$, where $k>0$. Compute the steady state solution (i.e., $\partial_t T = 0$) assuming the boundary conditions: $-k\partial_n T(a) = 1$, $T(b)=0$ (∂_n is the normal derivative).

At steady state, T does not depend on t and we have $\partial_{xx}T(x) = 0$, which implies $\partial_x T(x) = \alpha$, and $T(x) = \beta+\alpha x$, where $\alpha,\beta\in\mathbb{R}$. The two constants α and β are determined by the boundary conditions. $1 = -k\partial_n T(a) = k\partial_x T(a) = k\alpha$ and $0 = T(b) = \beta+\alpha b$. We conclude that $\alpha = \frac{1}{k}$ and $\beta = -\alpha b = -\frac{b}{k}$. In conclusion

$$T(x) = \frac{x-b}{k}.$$

Question 3: Consider the equation $\partial_t c(x,t) - \partial_x\big((1+x^2)\partial_x c(x,t)\big) = 6x/L^2$, where $x \in [0,L]$, $t > 0$, with $c(x,0) = f(x)$, $-\partial_n c(0,t) = 1$, $-\partial_n c(L,t) = \frac{2}{1+L^2}$, ($\partial_n$ is the normal derivative). Compute $E(t) := \int_0^L c(\xi,t)\mathrm{d}\xi$.

We integrate the equation with respect to x over $[0,L]$

$$\int_0^L \partial_t c(\xi,t)\mathsf{d}\xi - \int_0^L \partial_\xi\big((1+\xi^2)\partial_\xi c(\xi,t)\big)\mathsf{d}\xi = \frac{6}{L^2}\int_0^L \xi\mathsf{d}\xi.$$

Using that $\int_0^L \partial_t c(\xi,t)\mathsf{d}\xi = \mathsf{d}_t \int_0^T c(\xi,t)\mathsf{d}\xi$ together with the fundamental theorem of calculus, we infer that

$$\mathsf{d}_t E(t) - (1+L^2)\partial_x c(L,t) + \partial_x c(0,t) = 3.$$

The boundary conditions $\partial_x c(0,t) = -\partial_n c(0,t) = 1$, $-\partial_x c(L,t) = -\partial_n c(L,t) = \frac{2}{1+L^2}$ give

$$\mathsf{d}_t E(t) + 2 + 1 = 3.$$

We now apply the fundamental theorem of calculus with respect to t

$$E(t) - E(0) = \int_0^t \partial_\tau E(\tau)\mathsf{d}\tau = 0.$$

In conclusion

$$E(t) = \int_0^L f(\xi)\mathsf{d}\xi, \qquad \forall t \geq 0.$$

Question 4: Let $\phi = x^2 + 2y^2$ (a) Compute $\Delta\phi(x,y)$. (b) Consider the disk of radius 1 centered at $(0,0)$ and let Γ be the boundary of Ω. Compute $\int_\Gamma \partial_n\phi \mathrm{d}\Gamma$.

(a) The definition $\Delta\phi = \partial_{xx}\phi + \partial_{yy}\phi$ implies that

$$\Delta\phi = \partial_{xx}\phi + \partial_{yy}\phi = 2 + 4 = 6.$$

(b)The definition $\Delta\phi = \mathsf{div}(\nabla\phi)$ and the fundamental theorem of calculus (also known as the divergence theorem) imply that

$$\int_\Gamma \partial_n\phi\mathsf{d}\Gamma = \int_\Gamma n\cdot\nabla\phi\mathsf{d}\Gamma = \int_\Omega \mathsf{div}(\nabla\phi)\mathsf{d}\Omega = \int_\Omega \Delta\phi\mathsf{d}\Omega = 6\int_\Omega \mathsf{d}\Omega = 6\pi,$$

because the surface of Ω, $\int_\Omega \mathsf{d}\Omega$, is equal to π.