Quiz 1 (Notes, books, and calculators are not authorized)

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1: Let $\phi(x,y) = \log(3+\cos(x-y))$. Compute $\partial_x\phi(x,y)$ and $\partial_y\phi(x,y)$.

We apply the chain rule repeatedly

$$\partial_x\phi(x,y) = \frac{-1}{3+\cos(x-y)}\sin(x-y)$$

$$\partial_y\phi(x,y) = \frac{1}{3+\cos(x-y)}\sin(x-y).$$

Question 2: Let $\mathbf{f} \in C^1(\mathbb{R}^3;\mathbb{R}^3)$ be defined by $f(\mathbf{x}) = (\sin(x_1), \cos(x_1)x_2, x_1^2 - x_3^3)$. Compute $\text{div}(\mathbf{f})$.

We have

$$\begin{aligned}\mathsf{div}(\mathbf{f}) &= \partial_{x_1}(\sin(x_1)) + \partial_{x_2}(\cos(x_1)x_2) + \partial_{x_3}(x_1^2 - x_3^3)\\ &= \cos(x_1) + \cos(x_1) - 3x_3^2 = 2\cos(x_1) - 3x_3^2.\end{aligned}$$

Question 3: Let $\phi(x,y) = \cos(x)\sinh(y) + 2x^2 - 2xy - y^2$ (a) Compute $\Delta\phi(x,y)$.

The definition $\Delta\phi = \partial_{xx}\phi + \partial_{yy}\phi$ implies that

$$\Delta\phi = \partial_{xx}\phi + \partial_{yy}\phi = -\cos(x)\sinh(y) + \cos(x)\sinh(y) + 4 - 2 = 2.$$

(b)Let Ω be the disk of radius 1 centered at $(0,0)$ and let Γ be the boundary of Ω. Compute $\int_\Gamma \partial_n\phi \mathrm{d}\Gamma$.

The definition $\Delta\phi = \mathsf{div}(\nabla\phi)$ and the fundamental theorem of calculus (also known as the divergence theorem) implies that

$$\int_\Gamma \partial_n\phi \mathsf{d}\Gamma = \int_\Gamma n{\cdot}\nabla\phi \mathsf{d}\Gamma = \int_\Omega \mathsf{div}(\nabla\phi)\mathsf{d}\Omega = \int_\Omega \Delta\phi \mathsf{d}\Omega = 2\pi.$$

Hence $\int_\Gamma \partial_n\phi \mathsf{d}\Gamma = 2\pi$.

Question 4: Consider the heat equation $\partial_t T - k\partial_{xx} T = f(x)$, $x \in [0, L]$, $t > 0$, with $f(x) = -4k/L^2$, where $k > 0$. Compute the steady state solution (i.e., $\partial_t T = 0$) assuming the boundary conditions: $T(0) = -1$, $T(L) = 1$.

At steady state, T does not depend on t and we have $\partial_{xx}T(x) = 4/L^2$, which implies that

$$\partial_x T(x) = 4x/L^2 + \alpha,$$

and then

$$T(x) = \beta + \alpha x + 2x^2/L^2,$$

where $\alpha, \beta \in \mathbb{R}$. The two constants α and β are determined by the boundary conditions.

$$-1 = T(0) = \beta, \qquad 1 = T(L) = \beta + \alpha L + 2.$$

We conclude that $\alpha = 0$ and $\beta = -1$. In conclusion

$$T(x) = -1 + 2(x/L)^2.$$

Question 5: Consider the equation $\partial_t c(x,t) + \partial_x\big((x^3 - xL^2)c(x,t)\big) - \partial_x\big((1+2x^2)\partial_x c(x,t)\big) = 2x/L^2$, where $x \in [0, L]$, $t > 0$, with $c(x, 0) = f(x)$, $-\partial_n c(0,t) = 0$, $-\partial_n c(L,t) = \frac{1}{1+2L^2}$, ($\partial_n$ is the normal derivative). Compute $E(t) := \int_0^L c(\xi,t)\mathrm{d}\xi$. (*Hint:* Integrate the equation over $(0, L)$ and apply the fundamental theorem of calculus. There are many simplifications happening on the way.)

We integrate the equation with respect to x over $[0, L]$

$$\int_0^L \partial_t c(\xi,t)\mathsf{d}\xi + \int_0^L \partial_\xi\big((\xi^3 - \xi L^2)c(\xi,t)\big)\mathsf{d}\xi - \int_0^L \partial_\xi\big((1+2\xi^2)\partial_\xi c(\xi,t)\big)\mathsf{d}\xi = \frac{2}{L^2}\int_0^L \xi \mathsf{d}\xi.$$

Using that $\int_0^L \partial_t c(\xi,t)\mathsf{d}\xi = \partial_t \int_0^T c(\xi,t)\mathsf{d}\xi$ together with the fundamental theorem of calculus, we infer that

$$\partial_t E(t) - (1+2L^2)\partial_x c(L,t) + \partial_x c(0,t) = 1.$$

The boundary conditions $\partial_x c(0,t) = -\partial_n c(0,t) = 0$, $-\partial_x c(L,t) = -\partial_n c(L,t) = \frac{1}{1+2L^2}$ give

$$\partial_t E(t) + 1 = 1.$$

We now apply the fundamental theorem of calculus with respect to t

$$E(t) - E(0) = \int_0^t \partial_\tau E(\tau)\mathsf{d}\tau = 0.$$

In conclusion

$$E(t) = E(0) = \int_0^L c(\xi, 0)\mathsf{d}\xi = \int_0^L f(\xi)\mathsf{d}\xi, \qquad \forall t \geq 0.$$

Than it,

$$E(t) = \int_0^L f(\xi)\mathsf{d}\xi, \qquad \forall t \geq 0.$$
