Quiz 1 (Notes, books, and calculators are not authorized)

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1: Let $v : [0,\infty) \longrightarrow \mathbb{R}$ be a continuous function with bounded derivative, and let $w : [0,\infty) \longrightarrow \mathbb{R}$ be such that $w(x) = \frac{1}{x}\int_0^x (v(t) - v(x))\mathrm{d}t$. (a) Show that $|w(x)| \le \frac{Mx}{2}$ where $M = \sup_{x\in[0,\infty)} |\partial_x v(x)|$.

Any time we see a quantity like $v(t)-v(x)$ we must think of the fundamental theorem of calculus, i.e., $v(t)-v(x) = \int_x^t \partial_x v(z)\mathrm{d}z$. Hence, we have

$$|w(x)| = \frac{1}{x}\left|\int_0^x (v(t) - v(x))\mathrm{d}t\right| = \frac{1}{x}\left|\int_0^x \int_x^t \partial_z v(z)\mathrm{d}z\mathrm{d}t\right| \le \frac{1}{x}\int_0^x \left|\int_x^t \partial_z v(z)\mathrm{d}z\right| \mathrm{d}t \le \frac{1}{x}\int_0^x \int_t^x |\partial_z v(z)|\mathrm{d}z\mathrm{d}t$$
$$\le \frac{M}{x}\int_0^x \int_t^x \mathrm{d}z\mathrm{d}t = \frac{M}{x}\int_0^x (x-t)\mathrm{d}t = \frac{M}{x}(x^2 - \frac{1}{2}x^2) = \frac{M}{2}x.$$

Hence $|w(x)| \le \frac{Mx}{2}$ for all $x \in [0,\infty)$.

(b) Estimate $w(0)$.

The estimate $|w(x)| \le \frac{Mx}{2}$ shows that $|w(0)| \le 0$, meaning that $w(0) = 0$.

(c) Show that $\partial_t(tw(t)) = -t\partial_t v(t)$.

Upon observing that $tw(t) = \int_0^t (v(z) - v(t))\mathrm{d}z$ and recalling that the fundamental theorem of calculus implies that

$$\boxed{\partial_t \left(\int_0^t f(z)\mathrm{d}z\right) = f(t),}$$

we have

$$\partial(tw(t)) = \partial_t \int_0^t (v(z) - v(t))\mathrm{d}z = \partial_t \int_0^t v(z)\mathrm{d}z - \partial_t(v(t)t) = v(t) - v(t) - t\partial_t v(t) = -t\partial_t v(t).$$

Hence $\partial_t(tw(t)) = -t\partial_t v(t)$.

(d) Prove that $v(x) - v(0) = -w(x) - \int_0^x \frac{w(t)}{t}\mathrm{d}t$. (Hint: observe that $v(x) - v(0) = \int_0^x \frac{1}{t}(t\partial_t v(t))\,\mathrm{d}t$, use (c), and integrate by parts.)

We follow the hint

$$\begin{aligned} v(x) - v(0) &= \int_0^x \frac{1}{t}(t\partial_t v(t))\,\mathrm{d}t = -\int_0^x \frac{1}{t}\partial_t(tw(t))\mathrm{d}t \\ &= \int_0^x \partial_t\left(\frac{1}{t}\right) tw(t)\mathrm{d}t - \frac{1}{t}tw(t)\Big|_0^x \\ &= -\int_0^x \frac{1}{t^2}tw(t)\mathrm{d}t - w(x) + w(0), \end{aligned}$$

thereby proving that $v(x) - v(0) = -\int_0^x \frac{1}{t}w(t)\mathrm{d}t - w(x)$.

Question 2: Consider the equation $\partial_t c(x,t)+\partial_x\big((x^2-xL)c(x,t)\big)-\partial_x\big((1+x^2)\partial_x c(x,t)\big)=6x/L^2$, where $x\in[0,L]$, $t>0$, with $c(x,0)=f(x)$, $-\partial_n c(0,t)=2$, $-\partial_n c(L,t)=\frac{1}{1+L^2}$, ($\partial_n$ is the normal derivative). Compute $E(t):=\int_0^L c(\xi,t)\mathrm{d}\xi$.

We integrate the equation with respect to x over $[0,L]$

$$\int_0^L \partial_t c(\xi,t)\mathsf{d}\xi+\int_0^L \partial_x\big((x^2-xL)c(x,t)\big)\mathsf{d}\xi-\int_0^L \partial_\xi\big((1+\xi^2)\partial_\xi c(\xi,t)\big)\mathsf{d}\xi=\frac{6}{L^2}\int_0^L \xi\mathsf{d}\xi.$$

Using that $\int_0^L \partial_t c(\xi,t)\mathsf{d}\xi=\mathsf{d}_t\int_0^T c(\xi,t)\mathsf{d}\xi$ together with the fundamental theorem of calculus, we infer that

$$\mathsf{d}_t E(t)-(1+L^2)\partial_x c(L,t)+\partial_x c(0,t)=3.$$

The boundary conditions $\partial_x c(0,t)=-\partial_n c(0,t)=2$, $-\partial_x c(L,t)=-\partial_n c(L,t)=\frac{1}{1+L^2}$ give

$$\mathsf{d}_t E(t)+1+2=3.$$

We now apply the fundamental theorem of calculus with respect to t

$$E(t)-E(0)=\int_0^t \partial_\tau E(\tau)\mathsf{d}\tau=0.$$

In conclusion

$$E(t)=\int_0^L f(\xi)\mathsf{d}\xi,\qquad \forall t\geq 0.$$

Question 3: Let $\phi=\sin(x)\cosh(y)+2x^2+3y^2$ (a) Compute $\Delta\phi(x,y)$.

The definition $\Delta\phi=\partial_{xx}\phi+\partial_{yy}\phi$ implies that

$$\Delta\phi=\partial_{xx}\phi+\partial_{yy}\phi=-\sin(x)\cosh(y)+\sin(x)\cosh(y)4+6=10.$$

(b)Let Ω be the disk of radius 1 centered at $(0,0)$ and let Γ be the boundary of Ω. Compute $\int_\Gamma \partial_n\phi \mathrm{d}\Gamma$.

The definition $\Delta\phi=\mathsf{div}(\nabla\phi)$ and the fundamental theorem of calculus (also known as the divergence theorem) implies that

$$\int_\Gamma \partial_n\phi\mathsf{d}\Gamma=\int_\Gamma n{\cdot}\nabla\phi\mathsf{d}\Gamma=\int_\Omega \mathsf{div}(\nabla\phi)\mathsf{d}\Omega=\int_\Omega \Delta\phi\mathsf{d}\Omega=10\int_\Omega \mathsf{d}\Omega=10\pi,$$

because the surface of Ω, $\int_\Omega \mathsf{d}\Omega$, is equal to 10π.