Quiz 2 (Notes, books, and calculators are not authorized)

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1: Let ϕ be a non-zero solution to the eigenvalue problem $-\partial_{xx}\phi(x) = \lambda\phi(x)$, $x \in (0,\pi)$, $\phi(0) = 0$, $\partial_x\phi(\pi) + \phi(\pi) = 0$. Determine the sign of λ using the energy method.

Multiply the equation by ϕ, integrate over $(0,\pi)$, and apply the fundamental theorem of calculus (i.e. integrate by parts):

$$\lambda \int_0^\pi (\phi(x))^2 \mathsf{d}x = -\int_0^\pi \phi(x)\partial_{xx}\phi(x)\mathsf{d}x = -\int_0^\pi (\partial_x(\phi(x)\partial_x\phi(x)) - (\partial_x\phi(x))^2)\mathsf{d}x$$
$$= -\phi(\pi)\partial_x\phi(\pi) + \phi(0)\partial_x\phi(0) + \int_0^\pi (\partial_x\phi(x))^2\mathsf{d}x$$
$$= (\phi(\pi))^2 + \int_0^\pi (\partial_x\phi(x))^2\mathsf{d}x.$$

In conclusion

$$(\phi(\pi))^2 + \int_0^\pi (\partial_x\phi(x))^2\mathsf{d}x = \lambda \int_0^\pi (\phi(x))^2\mathsf{d}x.$$

Assuming that ϕ is nonzero, we obtain that $\lambda = ((\phi(\pi))^2 + \int_0^\pi (\partial_x\phi(x))^2\mathsf{d}x)/\int_0^\pi (\phi(x))^2\mathsf{d}x \geq 0$, i.e. λ is non-negative. If $\lambda = 0$ then $\phi(\pi) = 0$ and $\partial_x\phi = 0$, which implies that ϕ is constant. The other condition $\phi(\pi) = 0$ implies that $\phi = 0$ which contradicts our assumption that ϕ is non-zero. In conclusion λ is positive.

Question 2: Let $k, f : [-1,+1] \longrightarrow \mathbb{R}$ be such that $k(x) = 3$, $f(x) = -6$ if $x \in [-1,0]$ and $k(x) = 1$, $f(x) = 2$ if $x \in (0,1]$. Consider the boundary value problem $-\partial_x(k(x)\partial_x T(x)) = f(x)$ with $T(-1) = 1$ and $\partial_x T(1) = 1$.
(a) What should be the interface conditions at $x = 0$ for this problem to make sense?

The function T and the flux $k(x)\partial_x T(x)$ must be continuous at $x = 0$. Let T^- denote the restriction of the solution on $[-1,0]$ and T^+ be the restriction of the solution on $[0,+1]$. One should have

$$T^-(0) = T^+(0), \quad \text{and} \quad k^-(0)\partial_x T^-(0) = k^+(0)\partial_x T^+(0),$$

where $k^-(0) = 3$ and $k^+(0) = 1$.

(b) Solve the problem, i.e., find T sth. $-\partial_x(k(x)\partial_x T(x)) = f(x)$ with $T(-1) = 1$ and $\partial_x T(1) = 1$. Give all the details.

On the interval $[-1, 0]$ we have $k^-(x) = 3$ and $f^-(x) = -6$ which implies $-3\partial_{xx}T^-(x) = -6$. This in turn implies $T^-(x) = x^2 + ax + b$. The Dirichlet condition at $x = -1$ implies that $T^-(-1) = 1 = 1 - a + b$. This gives $a = b$ and $T^-(x) = x^2 + bx + b$.

We proceed similarly on the interval $[0, +1]$ and we obtain $-\partial_{xx}T^+(x) = 2$, which implies that $T^+(x) = -x^2 + cx + d$. The Neumann condition at $x = 1$ implies $\partial_x T^+(1) = 1 = -2 + c$. This gives $c = 3$ and $T^+(x) = -x^2 + 3x + d$.

The interface conditions $T^-(0) = T^+(0)$ and $k^-(0)\partial_x T^-(0) = k^+(0)\partial_x T^+(0)$ give $b = d$ and $3b = 3$, respectively. In conclusion $b = 1$, $d = 1$ and

$$T(x) = \begin{cases} x^2 + x + 1 & \text{if } x \in [-1, 0], \\ -x^2 + 3x + 1 & \text{if } x \in [0, 1]. \end{cases}$$

Question 3: Assume that the following equation has a smooth solution: $-\partial_x((1+x^2)\partial_x T(x)) + \partial_x T(x) + T(x) = \cos(x)$, $T(a) = 1$, $T(b) = \pi$, $x \in [a, b]$, $t > 0$, where $k > 0$. Prove that this solution is unique by using the energy method. (*Hint:* Do not try to simplify $-\partial_x((1 + x^2)\partial_x T)$.

Assume that there are two solutions T_1 and T_2. Let $\phi = T_2 - T_1$. Then

$$-\partial_x((1 + x^2)\partial_x \phi(x)) + \partial_x \phi(x) + \phi(x) = 0, \quad \phi(a) = 0, \quad \phi(b) = 0$$

Multiply the PDE by ϕ, integrate over (a, b), and integrate by parts (i.e., apply the fundamental theorem of calculus):

$$\begin{aligned} 0 &= \int_a^b \big(- \partial_x((1 + x^2)\partial_x \phi(x))\phi(x) + (\partial_x \phi(x))\phi(x) + (\phi(x))^2\big) \mathrm{d}x \\ &= \int_a^b \big(- \partial_x(\phi(x)(1 + x^2)\partial_x \phi(x)) + (1 + x^2)(\partial_x \phi(x))^2 + \partial_x(\frac{1}{2}\phi(x)^2) + (\phi(x))^2\big) \mathrm{d}x \\ &= \int_a^b \big((1 + x^2)(\partial_x \phi(x))^2 + (\phi(x))^2\big) \mathrm{d}x \end{aligned}$$

This implies $\int_a^b (\phi(x))^2 \mathrm{d}x = 0$, i.e., $\phi = 0$, meaning that $T_2 = T_1$.