Quiz 3 (Notes, books, and calculators are not authorized)

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1: Does any of the following expressions solve the Laplace equation inside the rectangle $0 \le x \le L$, $0 \le y \le H$, with the following boundary conditions $u(0,y) = 0$, $u(L,y) = 3\sinh(\frac{5\pi L}{H})\sin(\frac{5\pi y}{H})$, $u(x,0) = 0$, $u(x,H) = 0$? (justify clearly your answer):

$$u_1(x,y) = 3\cos(\frac{5\pi y}{H})\cosh(\frac{5\pi(x-L)}{H}), \quad u_2(x,y) = 3\sin(\frac{5\pi y}{H})\cosh(\frac{5\pi(x-L)}{H}),$$
$$u_3(x,y) = 3\cos(\frac{5\pi y}{H})\sinh(\frac{5\pi x}{H}), \quad u_4(x,y) = 3\sin(\frac{5\pi y}{H})\sinh(\frac{5\pi x}{H}).$$

From class, we know that all the above expressions solve the Laplace equation, hence we just need to verify that the boundary conditions are met. We observe that u_1 and u_3 do not satisfy the Dirichlet boundary conditions $u(x,0) = 0$, $u(x,H) = 0$; therefore u_1 and u_3 must be discarded.

Both u_2 and u_4 satisfy that Dirichlet conditions: $u_2(x,0) = 0$, $u_2(x,H) = 0$, and $u_4(x,0) = 0$, $u_4(x,H) = 0$. Now we need to check the Neumann conditions.

Note that u_2 is such that $u_2(0,y) = 3\sin(\frac{5\pi y}{H})\cosh(\frac{5\pi(-L)}{H}) \neq 0$, which shows that u_2 is not the solution to our problem either.

Finally $u_4(0,y) = 3\sin(\frac{5\pi y}{H})\sinh(0) = 0$ and $u_4(L,y) = 3\sin(\frac{5\pi y}{H})\sinh(\frac{5\pi L}{H})$; a result u_4 is the solution.

Question 2: The solution of the equation, $\frac{1}{r}\partial_r(r\partial_r u) + \frac{1}{r^2}\partial_{\theta\theta}u = 0$, inside the domain $D = \{\theta \in [0, \frac{\pi}{2}],\ r \in [0,2]\}$, subject to the boundary conditions $\partial_\theta u(r,0) = 0$, $u(r,\frac{\pi}{2}) = 0$, $u(2,\theta) = g(\theta)$ is $u(r,\theta) = \sum_{n=0}^{\infty} a_{2n+1}r^{(2n+1)}\cos((2n+1)\theta)$. What is the solution corresponding to $g(\theta) = 5\cos(3\theta) + 2\cos(5\theta)$? (Give all the details.)

The only non-zero terms in the expansion are $a_3r^3\cos(3\theta) + a_5r^5\cos(5\theta)$. The boundary condition $u(2,\theta) = 5\cos(3\theta) + 2\cos(5\theta) = a_3 2^3\cos(3\theta) + a_5 2^5\cos(5\theta)$ is satisfied if $a_3 = 5/(2^3)$ and $a_5 = 2/(2^5)$, i.e.,

$$u(r,\theta) = 5\frac{r^3}{2^3}\cos(3\theta) + 2\frac{r^5}{2^5}\cos(5\theta).$$

Question 3: Consider the triangular domain $D = \{(x, y); x \geq 0,\ y \geq 0, 1 - x - y \leq 0\}$. Let $f(x, y) = x^2 - y^2 - 3$. Let $u \in \mathcal{C}^2(D) \cap \mathcal{C}^0(\overline{D})$ solve $-\Delta u = 0$ in D and $u|_{\partial D} = f$. Compute $\min_{(x,y)\in\overline{D}} u(x, y)$ and $\max_{(x,y)\in\overline{D}} u(x, y)$.

We use the maximum principle (u is harmonic and has the required regularity). Then

$$\min_{(x,y)\in\overline{D}} u(x, y) = \min_{(x,y)\in\partial D} f(x, y), \quad \text{and} \quad \max_{(x,y)\in\overline{D}} u(x, y) = \max_{(x,y)\in\partial D} f(x, y).$$

A point (x, y) is at the boundary of D if and only if $\{x = 0$ and $y \in [0, 1]\}$ or $\{y = 0$ and $x \in [0, 1]\}$, or $\{1 - y - x = 0$ and $x \in [0, 1]\}$.
(i) In the first case, $x = 0$ and $y \in [0, 1]$, we have

$$f(x, y) = -y^2 - 3, \qquad y \in [0, 1].$$

The maximum is -3 and the minimum is -4.
(ii) In the second case, $y = 0$ and $x \in [0, 1]$, we have

$$f(x, y) = x^2 - 3, \qquad x \in [0, 1].$$

The maximum is -2 and the minimum is -3.
(iii) In the third case, $1 - x = y$ and $x \in [0, 1]$, we have

$$f(x, y) = x^2 - (1 - x)^2 - 3 = 2x - 4, \qquad x \in [0, 1].$$

The maximum is -2 and the minimum is -4.
We finally can conclude

$$\min_{(x,y)\in\partial D} f(x, y) = -4, \quad \max_{(x,y)\in\partial D} f(x, y) = -2.$$

In conclusion

$$\min_{(x,y)\in\overline{D}} u(x, y) = -4, \quad \max_{(x,y)\in\overline{D}} u(x, y) = -2$$

Question 4: Let Ω be an open connected set in $\mathbb{R}^2$. Let u be a real-valued nonconstant function continuous on $\overline{\Omega}$ and harmonic on Ω and of class $\mathcal{C}^2$ in Ω. Assume that there exists x_0 in Ω such $\nabla u(x_0) = 0$. Is the point x_0 a minimum, a maximum, or a saddle point? (explain)

The keyword here is that Ω is open, meaning that the points at the boundary of Ω are not in Ω (it is not possible to punch a hole around the boundary points). The point x_0 is in Ω, that is x_0 is not at the boundary. Since u is continuous on $\overline{\Omega}$, harmonic on Ω and of class $\mathcal{C}^2$ in Ω, the maximum principle can be applied. The Maximum principle implies that u cannot have either a minimum or a maximum at x_0. This means that x_0 is a saddle point.