Quiz 3 (Notes, books, and calculators are not authorized)

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1: Solve the PDE (note that the width and the height of the rectangle are not equal)

$$\begin{aligned} &\partial_{xx}u + \partial_{yy}u = 0, && 0 < x < 1,\ 0 < y < 2, \\ &u(x,0) = 8\sin(9\pi x),\ u(x,2) = 0, && 0 < x < 1, \\ &u(0,y) = \sin(2\pi y), u(1,y) = 0, && 0 < y < 2. \end{aligned}$$

The method of separation of variables studied in class tells us that the solution is a sum of terms like $\sin(n\pi x)\sinh(n\pi(y-2))$ and $\sin(m\pi y/2)\sinh(m\pi(x-1)/2)$. By looking at the boundary conditions we infer that there are two nonzero terms in the expansion: one corresponding to $n = 9$ and one corresponding to $m = 4$. This gives

$$u(x,y) = 8\sin(9\pi x)\frac{\sinh(9\pi(2-y))}{\sinh(18\pi))} + \sin(2\pi y)\frac{\sinh(2\pi(1-x))}{\sinh(2\pi))}$$

Question 2: The solution of the equation, $\frac{1}{r}\partial_r(r\partial_r u) + \frac{1}{r^2}\partial_{\theta\theta}u = 0$, inside the domain $D = \{\theta \in [0, \frac{\pi}{2}],\ r \in [0,3]\}$, subject to the boundary conditions $\partial_\theta u(r,0) = 0$, $u(r,\frac{\pi}{2}) = 0$, $u(3,\theta) = g(\theta)$ is $u(r,\theta) = \sum_{n=0}^{\infty} a_{2n+1} r^{(2n+1)} \cos((2n+1)\theta)$. What is the solution corresponding to $g(\theta) = 5\cos(\theta) + 2\cos(3\theta)$? (Give all the details.)

The only non-zero terms in the expansion are $a_1 r\cos(\theta) + a_3 r^3 \cos(3\theta)$. The boundary condition $u(3,\theta) = 5\cos(\theta) + 2\cos(3\theta) = a_1 3^1 \cos(3\theta) + a_3 3^3 \cos(5\theta)$ is satisfied if $a_1 = 5/3$ and $a_3 = 2/(3^3)$, i.e.,

$$u(r,\theta) = 5\frac{r}{3}\cos(\theta) + 2\frac{r^3}{3^3}\cos(3\theta).$$

Question 3: Consider the elliptic domain $D = \{(x,y); x^2+2y^2 \le 2\}$. Let $f(x,y) = x^2 - y^2 - 3$. Let $u \in \mathcal{C}^2(D) \cap \mathcal{C}^0(\overline{D})$ solve $-\Delta u = 0$ in D and $u|_{\partial D} = f$. Compute $\min_{(x,y)\in\overline{D}} u(x,y)$ and $\max_{(x,y)\in\overline{D}} u(x,y)$.

We use the maximum principle (u is harmonic and has the required regularity). Then

$$\min_{(x,y)\in\overline{D}} u(x,y) = \min_{(x,y)\in\partial D} f(x,y), \quad \text{and} \quad \max_{(x,y)\in\overline{D}} u(x,y) = \max_{(x,y)\in\partial D} f(x,y).$$

A point (x,y) is at the boundary of D if and only if $x^2+2y^2 = 2$, meaning that $x^2 = 2-2y^2$. In conclusion, for any point $(x,y) \in \partial D$ we have $f(x,y) = x^2 - y^2 - 3 = 2 - 2y^2 - y^2 - 3 = -1 - 3y^2$ and $y \in [-1,+1]$.
(1) The maximum of $-1-3y^2$ over the interval $[-1,+1]$ is reached for $y=0$. Hence $\max_{(x,y)\in\partial D} f(x,y) = -1$.
(2) The minimum of $-1-3y^2$ over the interval $[-1,+1]$ is reached for $y=\pm 1$. Hence $\min_{(x,y)\in\partial D} f(x,y) = -4$.
We finally can conclude

$$\min_{(x,y)\in\partial D} f(x,y) = -4, \qquad \max_{(x,y)\in\partial D} f(x,y) = -1.$$

In conclusion

$$\min_{(x,y)\in\overline{D}} u(x,y) = -4, \qquad \max_{(x,y)\in\overline{D}} u(x,y) = -1.$$

Question 4: Let $\Omega \subset \mathbb{R}^2$ be the disk centered at $(0,0)$ and of radius 1. Give an example of a nonzero smooth function that is harmonic in Ω and has a zero gradient at $(0,0)$.

Consider the function $u(x,y) = x^2 - y^2$. Clearly $\Delta u(x,y) = 0$ and $\nabla u(0,0) = (0,0)$.

Similarly $u(x,y) = 1$ satisfies all the requirements.