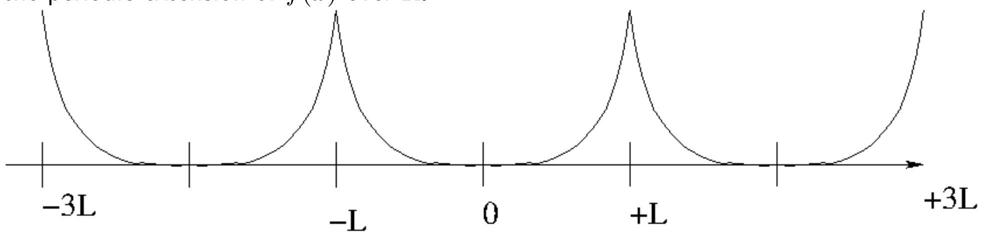
Quiz 4 (Notes, books, and calculators are not authorized)

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1: Consider $f : [-L, L] \longrightarrow \mathbb{R}$, $f(x) = x^4$. (a) Sketch the graph of the Fourier series of f and the grah of f.

$FS(f)$ is equal to the periodic extension of $f(x)$ over $\mathbb{R}$.



Question 2: (a) Compute the coefficients of the sine series of $f(x) = x$ for $x \in [0, +\pi]$. (Recall that by definition $\text{SS}(f)(x) = \sum_{m=1}^{+\infty} b_m \sin(mx)$ with $b_m = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(mx)\mathrm{d}x$.)

The definition of $\text{SS}(f)(x)$ implies that

$$\begin{aligned} b_m &= \frac{2}{\pi}\int_0^{\pi} x\sin(mx)\mathrm{d}x = -\frac{2}{\pi}\int_0^{\pi} -\frac{1}{m}\cos(mx)\mathrm{d}x + \frac{2}{\pi}[-x\frac{1}{m}\cos(mx)]_0^{\pi} \\ &= \frac{2}{m}(-1)^{m+1}. \end{aligned}$$

As a result $\text{SS}(f)(x) = \sum_{m=1}^{+\infty} \frac{2}{m}(-1)^{m+1}\sin(mx)$.

(b) For which values of x in $[0, +\pi]$ does the sine series coincide with $f(x)$? (Explain).

The sine series coincides with the function $f(x)$ over the entire interval $[0, +\pi)$ since $f(0) = 0$ and f is smooth over $[0, +\pi)$. The series does not coincide with $f(+\pi)$ since $f(+\pi) \neq 0$.

Question 3: Let L be a positive real number. Let $\mathbb{P}_1 = \text{span}\{1, \cos(\pi t/L), \sin(\pi t/L)\}$ and consider the norm $\|f\|_{L^2} := \left(\int_{-L}^{L} f(t)^2 \text{d}t\right)^{\frac{1}{2}}$. (a) Compute the best approximation of $h(t) = 5\sin(\pi t/L) + 7\sin(3\pi t/L)$ in $\mathbb{P}_1$ with respect to the L^2-norm.

Recall that the best approximation of h in $\mathbb{P}_1$, say $\Pi_{\mathbb{P}_1}(h)$, is such that $\int_{-L}^{L}(h(t) - \Pi_{\mathbb{P}_1}(h))p(t)\text{d}t = 0$ for all $p \in \mathbb{P}_1$. The function $h(t) - 5\sin(\pi t/L) = 7\sin(3\pi t/L)$ is orthogonal to all the members of $\mathbb{P}_1$ since the functions $\cos(m\pi t/L)$ and $\sin(m\pi t/L)$ are orthogonal to both $\cos(n\pi t/L)$ and $\sin(n\pi t/L)$ for all $m \neq m$; as a result, the best approximation of h in $\mathbb{P}_1$ is $5\sin(\pi t/L)$. In conclusion $\Pi_{\mathbb{P}_1}(h) = 5\sin(\pi t/L)$.

(b) Compute the best approximation of $2 - 3t + \cos(2\pi t/L)$ in $\mathbb{P}_1$ with respect to the L^2-norm. (Hint: $\int_{-L}^{L} t\sin(\pi t/L)\text{d}t = 2L^2/\pi$.)

Since $\cos(2\pi t/L)$ is othogonal to $\mathbb{P}_1$, the best approximation of $2 - 3t + \cos(2\pi t/L)$ in $\mathbb{P}_1$ is the same as that of $2 - 3t$. hence we just compute the best approximation of $2 - 3t$. We know from class that the truncated Fourier series

$$FS_1(2 - 3t + \cos(2\pi t/L)) = a_0 + a_1\cos(\pi t/L) + b_1\sin(\pi t/L)$$

is the best approximation. Now we compute a_0, a_1, a_2

$$\begin{aligned}
a_0 &= \frac{1}{2L}\int_{-L}^{L}(2 - 3t)\text{d}t = 2, \\
a_1 &= \frac{1}{L}\int_{-L}^{L}(2 - 3t)\cos(\pi t/L)\text{d}t = 0 \\
b_1 &= \frac{1}{L}\int_{-L}^{L}(2 - 3t)\sin(\pi t/L)\text{d}t = \frac{1}{L}\int_{-L}^{L} -3t\sin(\pi t/L)\text{d}t = -6\frac{L}{\pi}.
\end{aligned}$$

As a result

$$FS_1(2 - 3t + \cos(2\pi t/L)) = 2 - \frac{6L}{\pi}\sin(\pi t/L)$$
