Quiz 5 (Notes, books, and calculators are not authorized)

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**. Here are some formulae that you may want to use:

$$\mathcal{F}(f)(\omega) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)e^{i\omega x}dx, \qquad \mathcal{F}^{-1}(f)(x) = \int_{-\infty}^{+\infty} f(\omega)e^{-i\omega x}d\omega, \tag{1}$$

$$\mathcal{F}(f*g)(\omega) = 2\pi\mathcal{F}(f)(\omega)\mathcal{F}(g)(\omega), \qquad \mathcal{F}(f(x-\beta))(\omega) = \mathcal{F}(f)(\omega)e^{i\omega\beta} \tag{2}$$

$$\mathcal{F}(e^{-\alpha|x|}) = \frac{1}{\pi}\frac{\alpha}{\omega^2+\alpha^2}, \qquad \mathcal{F}(\frac{2\alpha}{x^2+\alpha^2})(\omega) = e^{-\alpha|\omega|}, \qquad \sqrt{\frac{\pi}{\alpha}}\mathcal{F}(e^{-\frac{x^2}{4\alpha}}) = e^{-\alpha\omega^2}. \tag{3}$$

Question 1: (i) Let f be an integrable function over $(-\infty, +\infty)$. Prove that for all $a, b \in \mathbb{R}$, and for all $\xi \in \mathbb{R}$, $\mathcal{F}([e^{ibx}f(ax)])(\xi) = \frac{1}{a}\mathcal{F}(f)(\frac{\xi+b}{a})$.

The definition of the Fourier transform together with the change of variable $ax \longmapsto x'$ implies

$$\begin{aligned}\mathcal{F}[e^{ibx}f(ax)])(\xi) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(ax)e^{ibx}e^{i\xi x}dx \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(ax)e^{i(b+\xi)x}dx \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{a}f(x')e^{i\frac{(\xi+b)}{a}x'}dx' \\ &= \tfrac{1}{a}\mathcal{F}(f)(\tfrac{\xi+b}{a}).\end{aligned}$$

Question 2: Solve the following integral equation: $\int_{-\infty}^{+\infty} f(y)f(x-y)\mathrm{d}y - 2\pi\frac{4}{x^2+4} = 0$, $\forall x \in \mathbb{R}$.

This equation can be re-written using the convolution operator:

$$f*f - 2\pi\frac{4}{x^2+4} = 0.$$

We take the Fourier transform and use the convolution theorem (2) together with (3) to obtain

$$\begin{aligned}2\pi\mathcal{F}(f)^2 - 2\pi e^{-2|\omega|} &= 0 \\ \mathcal{F}(f)^2 - e^{-2|\omega|} &= 0 \\ \mathcal{F}(f) &= \pm e^{-|\omega|}\end{aligned}$$

Taking the inverse Fourier transform, we obtain two solutions

$$f(x) = \pm\frac{2}{x^2+1}.$$

Question 3: State the shift lemma (Do not prove).

Let f be an integrable function over $\mathbb{R}$ (in $L^1(\mathbb{R})$), and let $\beta \in \mathbb{R}$. Using the definitions above, the following holds:

$$\mathcal{F}(f(x-\beta))(\omega) = \mathcal{F}(f)(\omega)e^{i\omega\beta}, \qquad \forall \omega \in \mathbb{R}.$$

Question 4: Use the Fourier transform technique to solve $\partial_t u(x,t) + \cos(t)\partial_x u(x,t) + 2u(x,t) = 0$, $x \in \mathbb{R}$, $t > 0$, with $u(x,0) = u_0(x)$.

Applying the Fourier transform to the equation gives

$$\partial_t \mathcal{F}(u)(\omega,t) + \cos(t)(-i\omega)\mathcal{F}(u)(\omega,t) + 2\mathcal{F}(u)(\omega,t) = 0$$

This can also be re-written as follows:

$$\frac{\partial_t \mathcal{F}(u)(\omega,t)}{\mathcal{F}(u)(\omega,t)} = i\omega\cos(t) - 2.$$

Then applying the fundamental theorem of calculus we obtain

$$\log(\mathcal{F}(u)(\omega,t)) - \log(\mathcal{F}(u)(\omega,0)) = i\omega\sin(t) - 2t.$$

This implies

$$\mathcal{F}(u)(\omega,t) = \mathcal{F}(u_0)(\omega)e^{i\omega\sin(t)}e^{-2t}.$$

Then the shift lemma gives

$$\mathcal{F}(u)(\omega,t) = \mathcal{F}(u_0(x-\sin(t)))(\omega)e^{-2t}.$$

This finally gives

$$u(x,t) = u_0(x-\sin(t))e^{-2t}.$$
