Last name: ***name:***

HW 11 extra, not graded.

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1: Consider the equation $\partial_t u + \partial_x(u^4) = 0$, where $x \in (-\infty, +\infty)$, $t > 0$, with initial data $u_0(x) = 0$ if $x < 0$, $u_0(x) = x^{\frac{1}{3}}$ if $0 < x < 1$, and $u_0(x) = 0$ if $1 < x$.

(a) Give without proof the implicit representation of the solution given by the method of characteristics.

We have $f(v) = v^4$ and $f'(v) = 4v^3$. The implicit form of the solution given by the method of characteristics is

$$\begin{aligned}
&X(\tau, s) = f'(u_0(s))\tau + s, && \forall \tau > 0,\ s \in \mathbb{R},\\
&T(\tau, s) = \tau, && \forall \tau > 0,\ s \in \mathbb{R},\\
&u(X(\tau, s), T(\tau, s)) = u - 0(s), && \forall \tau > 0,\ s \in \mathbb{R}.
\end{aligned}$$

(b) Give the explicit representation of the solution without assuming that there is a shock.

We have $f(v) = v^4$ and $f'(v) = 4v^3$. Looking at the definition of u_0, we need to consider three cases:

Case 1 ($s = 0$): We have $u_0(s) = 0$, $f'(u_0(s)) = 0$. Let $(x, t) \in \mathbb{R}\times\mathbb{R}_{t>0}$. let us find (τ, s) so that $x = X(\tau, s)$ and $t = T(\tau, s)$. We have

$$x = s, \quad \text{if} \quad s < 0.$$

Hence

$$\boxed{u(x, t) = 0, \quad \text{if} \quad x < 0.}$$

Case 2 ($0 \le s \le 1$): We have $u_0(s) = s^{\frac{1}{3}}$, $f'(u_0(s)) = 4s$. Let $(x, t) \in \mathbb{R}\times\mathbb{R}_{t>0}$. let us find (τ, s) so that $x = X(\tau, s)$ and $t = T(\tau, s)$. We have $x = 4st + s$; hence, $s = \frac{x}{4t+1}$

$$s = \frac{x}{4t+1}, \quad \text{if} \quad 0 < s < 1.$$

Hence

$$\boxed{u(x, t) = \left(\frac{x}{4t+1}\right)^{\frac{1}{3}}, \quad \text{if} \quad 0 < x < 4t + 1.}$$

Case 3 ($1 < s$): We have $u_0(s) = 0$, $f'(u_0(s)) = 0$. Let $(x, t) \in \mathbb{R}\times\mathbb{R}_{t>0}$. let us find (τ, s) so that $x = X(\tau, s)$ and $t = T(\tau, s)$. We have $x = s$; hence,

$$s = x, \quad \text{if} \quad 1 < s.$$

Hence

$$\boxed{u(x, t) = 0, \quad \text{if} \quad 1 < x.}$$

(c) By invoking the characteristics of the flow, explain why the solution obtained in (b) is ambiguous.

The characteristics cross in the space-time region $\{(x,t) \in \mathbb{R}\times\mathbb{R}_{>0};\ 1 < x < 4t+1$. Hence, the solution given by the method of characteristics is not unique. This means that one must introduce a shock in this region to disambiguate the solution.

(d) Compute the speed of the shock without invoking the Rankine-Hugoniot condition, i.e., just use mass conservation. (*Hint:* Look at the lecture notes to see how this is done. If you did not solve (b) correctly, you can use that $u(x,t) = (x/(1+4t))^{\frac{1}{3}}$ for all $0 \le x \le x_s(t)$.)

Let $x_s(t)$ be the position of the shock. We know that $x_s(0) = 1$. The solution on the left of the shock is $u_s^-(t) = \left(\frac{x_s(t)}{4t+1}\right)^{\frac{1}{3}}$. The solution on the right of the shock is $u_s^+(t) = 0$. We compute the value of $u_s^-(t)$ by enforcing mass conservation. The total mass is conserved:

$$\int_{-\infty}^{\infty} u(x,t)\mathsf{d}x = \int_0^{x_s(t)} u(x,t)\mathsf{d}x = \int_0^{x_s(0)} u_0(x)\mathsf{d}x = \int_0^1 x^{\frac{1}{3}}\mathsf{d}x = \frac{3}{4}$$

i.e., using the fact that $u(x,t) = (x/(1+4t))^{\frac{1}{3}}$ for all $0 \le x \le x_s(t)$, we have

$$\frac{3}{4} = (1+4t)^{-\frac{1}{3}} \int_0^{x_s(t)} x^{\frac{1}{3}}\mathsf{d}x = (1+4t)^{-\frac{1}{3}}\frac{3}{4}x_s(t)^{\frac{4}{3}}.$$

This gives the shock speed

$$x_s(t) = (1+4t)^{\frac{1}{4}}.$$

(e) Use the Rankine-Hugoniot relation to compute the shock speed. (*Hint:* Look at the lecture notes to see how this is done. If you did not solve (b) correctly, you can use that $u(x,t) = (x/(1+4t))^{\frac{1}{3}}$ for all $0 \le x \le x_s(t)$.)

The solution is $u(x,t) = (\frac{x}{1+4t})^{\frac{1}{3}}$ on the left of the shock and $u(x,t) = 0$ on the right. The speed of the shock is given by the Rankin-Hugoniot formula

$$\frac{\mathrm{d}x_s(t)}{\mathrm{d}t} = \frac{(u_s^+(t))^4 - (u_s^-(t))^4}{u_s^+(t) - u_s^-(t)}, \qquad \text{and } x_s(0) = 1,$$

where $u_s^+(t) = 0$ and $u_s^-(t) = \left(\frac{x_s(t)}{1+4t}\right)^{\frac{1}{3}}$. This gives

$$\frac{\mathrm{d}x_s(t)}{\mathrm{d}t} = (u_s^-(t))^3 = \frac{x_s(t)}{1+4t},$$

which we re-write as follows:

$$\frac{\mathrm{d}\log(x_s(t))}{\mathrm{d}t} = \frac{1}{1+4t} = \frac{1}{4}\frac{\mathrm{d}\log(1+4t)}{\mathrm{d}t}.$$

Applying the fundamental of calculus between 0 and t gives

$$\log(x_s(t)) - \log(1) = \frac{1}{4}(\log(1+4t) - \log(1)).$$

This give

$$x_s(t) = (1+4t)^{\frac{1}{4}}.$$

(f) Give the final and complete expression of the solution including the presence of the moving shock

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The solution is finally expressed as follows:

$$u(x,t) = \begin{cases} 0 & \text{if } x < 0 \\ \left(\frac{x}{1+4t}\right)^{\frac{1}{3}} & \text{if } 0 < x < (1+4t)^{\frac{1}{4}} \\ 0 & \text{if } (1+4t)^{\frac{1}{4}} < x \end{cases}$$

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Question 2: Consider the equation $\partial_t v + \partial_x(-v^6) = 0$, where $x \in \mathbb{R}$, $t \in \mathbb{R}_{>0}$, with initial data $u_0(x) = 1$ if $x < 0$, $u_0(x) = 0$ otherwise. Let $u(x,t) := 1 - H(x+t)$. Our objective is to show that $u(x,t)$ is a solution to this problem in the distribution sense. We denote δ_a the Dirac measure supported at a, i.e., $\delta_a(\varphi) := \varphi(a)$ for all $\varphi \in C^0(\mathbb{R})$. Accept as a fact that the chain rule holds for distributions.

(a) Express u_0 in terms of the Heaviside function.

We have

$$u_0(x) = 1 - H(x) \qquad \forall x \in \mathbb{R}.$$

(b) Compute $\partial_t u(x,t)$.

Calling ξ the independent variable of H and using the chain rule, we have

$$\partial_t u(x,t) = -\partial_t(H(x+t)) = -(\partial_\xi H)(x+t)\partial_t(x+t) = -\delta_{-t}$$

Hence for all $t > 0$

$$\partial_t u(x,t) = -\delta_{-t}.$$

Meaning that for all $t > 0$

$$\langle \partial_t u(x,t), \varphi \rangle = -\varphi(-t) \qquad \forall \varphi \in C^0(\mathbb{R}).$$

(c) Compute $u(x,t)^6$.

We have

$$u(x,t)^6 = (1 - H(x+t))^6.$$

If $x + t < 0$, then $u(x,t)^6 = 1$. If $x + t > 0$, then $u(x,t)^6 = 0$. This proves that

$$u(x,t)^6 = 1 - H(x+t).$$

(c) Show that $u(x,t) = 1 - H(x+t)$ is a solution in the distribution sense.

Using that $(1 - H(x+t))^6 = 1 - H(x+t)$, we also have

$$\partial_x(u^6) = \partial_x(1 - H(x+t)) = -\partial_x H(x+t) = -\delta_{-t}.$$

This implies that

$$\partial_t u + \partial_x(-u^6) = -\delta_{-t} - (-\delta_{-t}) = 0.$$

Moreover, $u(x,0) = 1 - H(x) = u_0(x)$. In conclusion $u(x,t) = 1 - H(x+t)$ is a solution in the distribution sense.

(d) Let $x_0 < x_1$ be two real numbers and let $t > 0$. Let $u(x,t) := 1 - H(x+t)$. Compute $\partial_t \int_{x_0}^{x_1} u(x,t)\mathrm{d}x$.

Case 1: Assume that $x_1 + t < 0$. Then $x + t < x_1 + t < 0$ for all $x \in (x_0, x_1)$, whence $u(x,t) = 1$ for all $x \in (x_0, x_1)$. This shows that

$$\partial_t \int_{x_0}^{x_1} u(x,t)\mathsf{d}x = \partial_t (x_1 - x_0) = 0.$$

Case 2: Assume that $x_0 + t < 0 < x_1 + t$. Then

$$\int_{x_0}^{x_1} u(x,t) = \int_{x_0}^{-t} \mathsf{d}x = (-t - x_0).$$

Hence

$$\partial_t \int_{x_0}^{x_1} u(x,t)\mathsf{d}x = -1.$$

Case 3: Assume that $0 < x_0 + t$. Then $0 < x_0 + t < x + t$ for all $x \in (x_0, x_1)$, whence $u(x,t) = 0$ for all $x \in (x_0, x_1)$. This shows that

$$\partial_t \int_{x_0}^{x_1} u(x,t)\mathsf{d}x = 0.$$

(e) Let $x_0 < x_1$ be two real numbers and let $t > 0$. Let $u(x,t) := 1 - H(x+t)$. With $f(v) = -v^6$, compute

$$\partial_t \int_{x_0}^{x_1} u(x,t)\mathrm{d}x + f(u(x_1,t)) - f(u(x_0,t))$$

and relate your result to Definition 26.3 in the lecture notes.

Case 1: Assume that $0 < x_0 + t$ or $x_1 + t < 0$, Then $u(x_0,t) = u(x_1,t)$ and $\partial_t \int_{x_0}^{x_1} u(x,t)\mathsf{d}x = 0$. Hence,

$$\partial_t \int_{x_0}^{x_1} u(x,t)\mathsf{d}x + f(u(x_1,t)) - f(u(x_0,t)) = 0.$$

Case 2: Assume that $x_0 + t < 0$ and $0 < x_1 + t$. Then $u(x_0,t) = 1$ and $u(x_1,t) = 0$ and $\partial_t \int_{x_0}^{x_1} u(x,t)\mathsf{d}x = -1$. Hence,

$$\partial_t \int_{x_0}^{x_1} u(x,t)\mathsf{d}x + f(u(x_1,t)) - f(u(x_0,t)) = -1 - (-1^6) = -1 + 1 = 0.$$

We have thus establishes that for all $x_0 < x_1$ such that $x_0 \neq 1$ and $x_1 \neq t$

$$\partial_t \int_{x_0}^{x_1} u(x,t)\mathsf{d}x + f(u(x_1,t)) - f(u(x_0,t)) = 0.$$

That is to say the above equality holds for almost every $x_0 < x_1$ and $t > 0$. Hence u is a weak solution according Definition 26.3 in the lecture notes.

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Question 3: Consider the conservation equation with flux $q(\rho) = \rho^4$. Assume that the initial data is $\rho_0(x) = 2$, if $x < 0$, $\rho_0(x) = 1$, if $0 < x < 1$, and $\rho_0(x) = 0$, if $1 < x$.

(a) Recall the formulas for the characteristics (i.e., give the expressions of $X(\tau, s)$ and $T(\tau, s)$).

We recall that we have established in class that for initial value scalar conservation equations the characteristics are defined by

$$X(\tau, s) = \tau q'(\rho_0(s)) + s,$$
$$T(\tau, s) = \tau.$$

for all $\tau > 0$ and $s \in \mathbb{R}$.

(b) Draw the characteristics in the space-time plane (t vs. x)

We have

$$X(\tau, s) = T(\tau, s)q'(\rho_0(s)) + s.$$

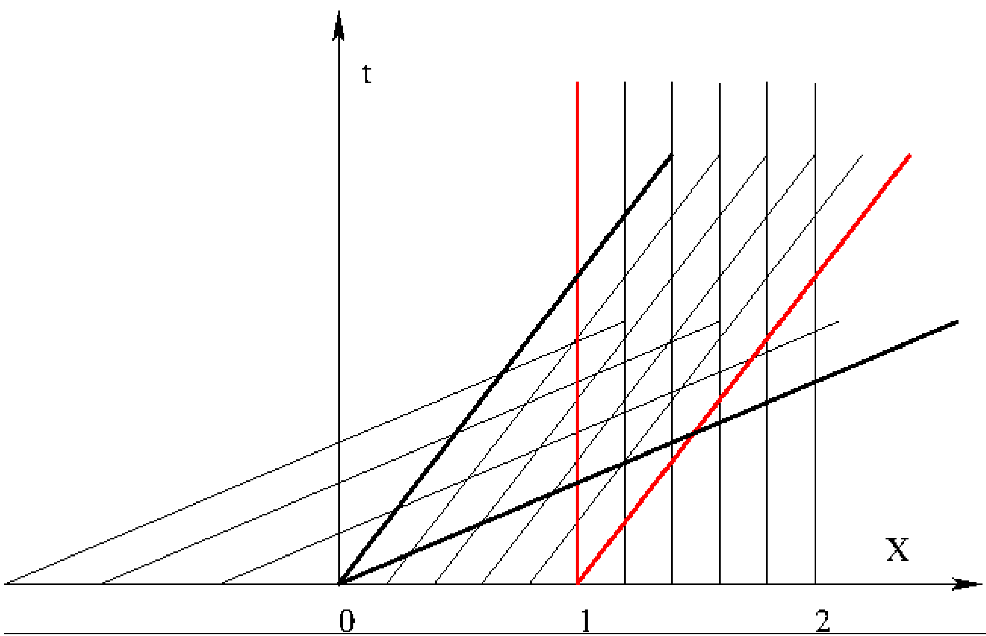
There are three families of characteristics depending on the value of $s \in \mathbb{R}$.

Case 1: $s < 0$, $X(s,t) = 32t + s$. In the x-t plane, these are lines with slope $\frac{1}{32}$.

Case 2: $0 < s < 1$, $X(s,t) = 4t + s$. In the x-t plane, these are lines with slope $\frac{1}{4}$.

Case 3: $1 < s$, $X(s,t) = s$. In the x-t plane, these are vertical lines.

One shock forms between the two black characteristics and another forms between the two red characteristic (see figure).



(c) Describes what happens by looking at the characteristics.

One shock forms between the two black characteristics and another forms between the two red characteristic (see figure). The leftmost shock travels at speed $\frac{2^4-1^4}{2-1} = 15$. The trajectory of this shock is $x_1(t) = 15t$. The second shock moves at speed $\frac{1^4-0}{1-0} = 1$. The trajectory of this shock is $x_2(t) = 1 + t$. This means that the leftmost shock catches the second when $15t = x_1(t) = x_2(t) = 1 + t$, i.e., the two shocks collide at time $\frac{1}{14}$ at the position $\frac{15}{14}$. After $t = \frac{1}{14}$, the two shocks combine and form a new shock moving at speed $\frac{2^4-0}{2-0} = 8$. The trajectory of the shock thereafter is $x_3(t) = \frac{15}{14} + 8t$.

Question 4: Assume $u_1 > u_2 \geq u_3 \geq 0$ and consider the following conservation equation

$$\partial_t u + \partial_x(\tfrac{1}{2}u^2) = 0, \quad x \in (-\infty, +\infty),\ t > 0, \qquad u(x,0) = u_0(x) := \begin{cases} 0 & \text{if } x \leq 0, \\ u_1 x & \text{if } 0 < x \leq 1, \\ u_1 & \text{if } 1 < x \leq 2, \\ u_2 & \text{if } 2 < x \leq 3, \\ u_3 & \text{if } 3 \leq x. \end{cases}$$

(i) Assume $u_2 = u_3$. Solve until the expansion catches up the shock. When does it happen?

Let us compute the explicit solution. Let $x \in \mathbb{R}$ and $t > 0$. We want to find τ and s so that $x = X(\tau, s)$ and $t = T(\tau, s)$ with

$$X(\tau, s) = u_0(s)\tau + s, \quad T(\tau, s) = \tau$$

Hence $\tau = t$. We now look for s.

Case 1: If $s \leq 0$, we have $u_0(s) = 0$ and $X(t,s) = s$; as a result, $s = X(t,s)$, and

$$u(x,t) = 0, \qquad \text{if } x \leq 0.$$

Case 2: If $0 < s \leq 1$, we have $u_0(s) = u_1 s$ and $X(t, x_0) = u_1 st + s$; as a result $s = X/(1 + u_1 t)$, and

$$u(x,t) = u_1 x/(1 + u_1 t), \qquad \text{if } 0 < x \leq 1 + u_1 t.$$

case 3: If $1 < s \leq 2$, we have $u_0(s) = u_1$ and $X(t, x_0) = u_1 t + s$; as a result $s = X(t,s) - u_1 t$, which implies

$$u(x,t) = u_1, \qquad \text{if } 1 + u_1 t < x \leq 2 + u_1 t.$$

Case 4: If $2 < s$, we have $u_0(s) = u_2$ and $X(t,s) = u_2 t + s$; as a result $s = X(t,s)$, which implies

$$u(x,t) = u_2 \qquad \text{if } 2 + u_2 t < x.$$

We have a shock at $x = 2$ and $t = 0$. The speed of the shock is given by the Rankin-Hugoniot formula

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \frac{\frac{1}{2}u_1^2 - \frac{1}{2}u_2^2}{u_1 - u_2} = \frac{1}{2}(u_1 + u_2).$$

As a result $x_1(t) = 2 + \frac{1}{2}(u_1 + u_2)t$. This implies that the solution is

$$u(x,t) = \begin{cases} 0, & \text{if } x \leq 0, \\ u_1 x/(1 + u_1 t), & \text{if } 0 < x \leq 1 + u_1 t, \\ u_1, & \text{if } 1 + u_1 t < x \leq 2 + \frac{1}{2}(u_1 + u_2)t, \\ u_2, & \text{if } 2 + \frac{1}{2}(u_1 + u_2)t < x. \end{cases}$$

The expansion wave catches up the shock when the interval $(1 + u_1 t, 2 + \frac{1}{2}(u_1 + u_2)t$ is empty. Hence the time T when the expansion wave catches up the shock is defined by

$$2 + \frac{1}{2}(u_1 + u_2)T = 1 + u_1 T,$$

that is to say

$$T = \frac{2}{u_1 - u_2}.$$

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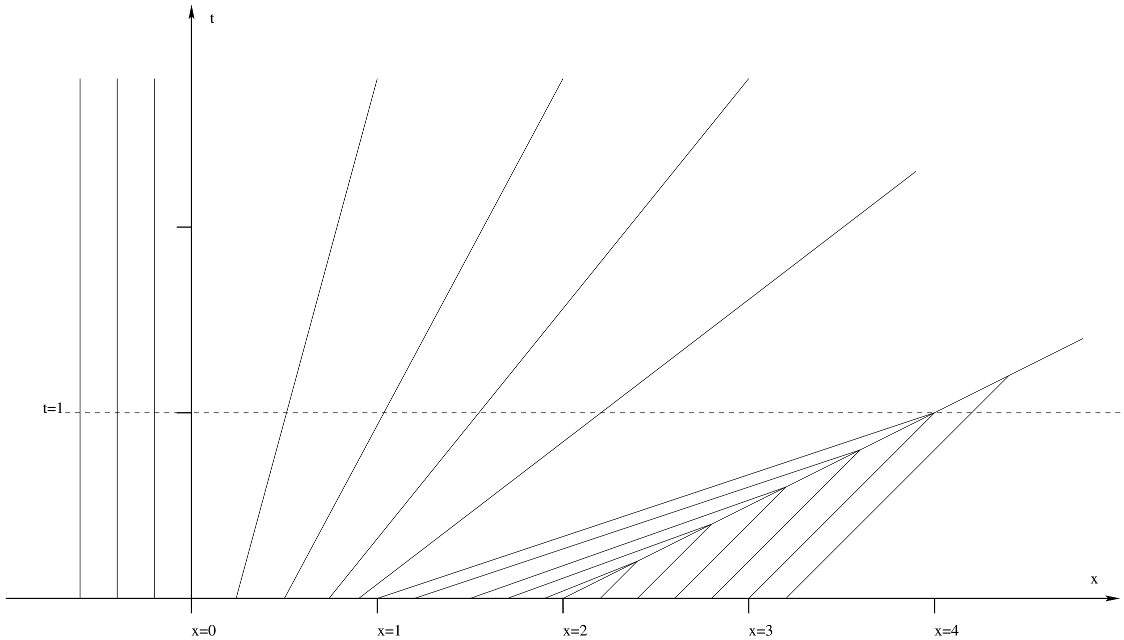
(ii) Draw the characteristics corresponding to the situation (i) with $u_1 = 2$ and $u_2 = 1$.



(iii) Assume now that $u_1 > u_2 > u_3 = 0$. When does the first shock catches the second one?

The speed of the first shock (starting at $x = 2$ when $t = 0$) is given by the Rankin-Hugoniot formula

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \frac{\frac{1}{2}u_1^2 - \frac{1}{2}u_2^2}{u_1 - u_2} = \frac{1}{2}(u_1 + u_2).$$

As a result $x_1(t) = 2 + \frac{1}{2}(u_1 + u_2)t$. The speed of the second shock (starting at $x = 3$ when $t = 0$) is given by the Rankin-Hugoniot formula

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \frac{\frac{1}{2}u_2^2}{u_2} = \frac{1}{2}u_2.$$

As a result $x_2(t) = 3 + \frac{1}{2}u_2 t$.

The time T' when the two shocks are at the same location is such that $x_1(T') = x_2(T')$; that is to say,

$$2 + \frac{1}{2}(u_1 + u_2)T' = 3 + \frac{1}{2}u_2 T',$$

which gives

$$T' = \frac{2}{u_1}.$$

Note that $T > T'$ for all $u_2 > 0$. This means that the first shock catches up the second one before the fans catches the first shock.