***Last name:*** ***name:***

HW 2 (Notes, books, and calculators are not authorized)

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

**Question 1:** Let $D = (-2, 1)$. Consider the boundary value problem $-\partial_x(k(x)\partial_x T(x)) = f(x)$ for $x \in D$ with the boundary conditions with $T(-2) = 0$, $\partial_x T(1) = -\pi$, and

$$f(x) = \begin{cases} -2\pi & x \in (-2, 0), \\ 2\pi^2 \sin(\pi x) & x \in [0, 1) \end{cases} \qquad k(x) = \begin{cases} \pi & x \in (-2, 0), \\ 2 & x \in [0, 1). \end{cases}$$

(i) What should be the interface conditions at $x = 0$ for this problem to make sense?

The function $T$ and the flux $k(x)\partial_x T(x)$ must be continuous at $x = 0$. One should have $T(0^-) = T(0^+)$ and $k(0^+)\partial_x T(0^-) = k(0^+)\partial_x T(0^+)$, where $k(0^-) = \pi$ and $k(0^+) = 2$.

(ii) Solve the problem.

On the interval $(-2, 0)$ we have $k(x) = \pi$ and $f(x) = -2\pi$; hence, $-\pi\partial_{xx} T = -2\pi$. Using the fundamental theorem twice we obtain

$$T(x) = x^2 + ax + b.$$

Using the boundary condition $T(-2) = 0$, we obtain $4 - 2a + b = 0$.

On the interval $[0, 1)$ we have $k(x) = 2$ and $f(x) = 2\pi^2 \sin(\pi x)$; hence, $-2\partial_{xx} T = 2\pi^2 \sin(\pi x)$. Using the fundamental theorem twice we obtain

$$T(x) = \sin(\pi x) + cx + d.$$

Using the boundary condition $\partial_x T(1) = -\pi$, we obtain $-\pi + c = -\pi$; hence, $c = 0$.

Now we enforce continuity of $T$ at $0$: $b = d$.

The continuity of the flux $k\partial T$ gives $\pi a = 2\pi$; hence, $a = 2$.

Finally we have $a = 2$, $b = 0$, $c = 0$, $d = 0$. Hence

$$\boxed{T(x) = \begin{cases} x^2 + 2x & x \in (-2, 0), \\ \sin(\pi x) & x \in [0, 1). \end{cases}}$$

**Question 2:** Let $k, f : (-1, 2) \longrightarrow \mathbb{R}$ be such that $k(x) = 2$, $f(x) = 0$ if $x \in (-1, 0)$ and $k(x) = 1$, $f(x) = 2$ if $x \in [0, 2)$. Find $T$ so that $-\partial_x(k(x)\partial_x T(x)) = f(x)$ with $T(-1) = -2$ and $T(2) = 2$.

On the interval $(-1, 0)$ we have $k(x) = 2$ and $f(x) = 0$ which implies $-\partial_{xx} T(x) = 0$. This in turn implies $T(x) = ax + b$. The Dirichlet condition at $x = -1$ implies that $T(-1) = -2 = -a + b$. This gives $a = b + 2$ and $T(x) = (b + 2)x + b$ for all $x \in (-1, 0)$.

We proceed similarly on the interval $[0, +2]$ and we obtain $-\partial_{xx} T(x) = 2$, which implies that $T(x) = -x^2 + cx + d$. The Dirichlet condition at $x = 2$ implies $T(2) = 2 = -4 + 2c + d$. This gives $c = 3 - \frac{d}{2}$ and $T(x) = -x^2 + (3 - \frac{d}{2})x + d$ for all $x \in [0, 1)$.

The interface conditions $T(0^-) = T(0^+)$ and $k(0^-)\partial_x T(0^-) = k(0^+)\partial_x T(0^+)$ give $b = d$ and $2(b + 2) = 3 - \frac{d}{2}$, respectively. In conclusion $b = -\frac{2}{5}$, $d = -\frac{2}{5}$ and

$$\boxed{T(x) = \begin{cases} \frac{8}{5}x - \frac{2}{5} & \text{if } x \in (-1, 0), \\ -x^2 + \frac{16}{5}x - \frac{2}{5} & \text{if } x \in [0, 2). \end{cases}}$$

**Question 3:** Consider the following problem: For $\alpha, \beta \in \mathbb{R}$, find $u$ in $C^2([0,1];\mathbb{R})$ so that $\partial_{xx}u = 0$ and $\partial_x u(0) = \alpha$, $\partial_x u(1) = \beta$.
(a) Prove that this problem is solvable if and only if $\alpha = \beta$.

(i) Assume that the problem is solvable. Let $u \in C^2([0,1];\mathbb{R})$ be a solution. Then

$$0 = \int_0^1 \partial_{xx} u \mathrm{d}x = \partial_x u(1) - \partial_x u(0) = \beta - \alpha.$$

This proves that $\alpha = \beta$.

(ii) Conversely, let us assume that $\alpha = \beta$. The equation $\partial_{xx}u(x) = 0$, $x \in (0,1)$, implies that $u(x) = ax + b$ for some $a, b \in \mathbb{R}$. The boundary conditions imply that

$$\partial_x u(0) = a = \alpha, \quad \partial_x u(1) = a = \beta = \alpha.$$

Hence $u(x) = \alpha x + b$, where $b$ is arbitrary. As $u$ is clearly in $C^2([0,1];\mathbb{R})$, this proves that the problem is solvable.

(b) Assume $\alpha = \beta = 1$. Is the solution to the problem unique?

We have established above that the solution is $u(x) = x + b$ where $b$ is arbitrary. Hence the solution is not unique.

(c) Is the problem well-posed in Hadamard's sense for all $(\alpha, \beta) \in \mathbb{R}$?

The problem is not well-posed in Hadamard's sense because there are no solution if $\alpha \neq \beta$.

**Question 4:** Let $D$ be a domain in $\mathbb{R}^d$ with a smooth boundary. Let $g : \partial D \to \mathbb{R}$ and assume that $g \in C^0(\partial D;\mathbb{R})$. Give one condition on $g$ so that the following problem cannot have a solution: Find $u \in C^2(D)$ so that $\Delta u(\mathbf{x}) = 0$ for all $\mathbf{x} \in D$ and $\partial_{\mathbf{n}} u(\mathbf{x}) = g(\mathbf{x})$ for all $\mathbf{x} \in \partial D$. (*Hint:* Think of the divergence theorem)

Applying the fundamental divergence theorem we obtain

$$0 = \int_D \Delta u \mathrm{d}x = \int_D \nabla\cdot\nabla u \mathrm{d}x = \int_{\partial D} \partial_{\mathbf{n}} u(\mathbf{x}) \mathrm{d}s.$$

Hence, the problem is not solvable if $\int_{\partial D} g(\mathbf{x}) \mathrm{d}s \neq 0$.

**Question 5:** Consider the following problem: For all $(\alpha, \beta) \in \mathbb{R}^2$, find $u$ in $C^2([0, L]; \mathbb{R})$ so that $\partial_{xx} u = 0$ and $u(0) = \alpha$, $u(L) = \beta$. We set $G := \mathbb{R}^2$ and define the norm in $G$ to be $\|(\alpha, \beta)\|_G = \max(|\alpha|, |\beta|)$. We set $U := C^2([0, 1]; \mathbb{R})$ and define the norm $\|u\|_U := \max_{x \in D}(L^2|\partial_{xx} u| + L|\partial_x u| + |u(x)|)$. Prove that this problem is well-posed in the Hadamard sense.

As $\partial_{xx} u = 0$, the solution must be of the form $u(x) = ax + b$. The boundary conditions imply that $\alpha = b$ and $\beta = aL + b$. Hence $a = \frac{\beta - \alpha}{L}$, $b = \alpha$, and

$$u(x) = \frac{\beta - \alpha}{L} x + \alpha = \beta \frac{x}{L} + \alpha \frac{L - x}{L}.$$

This proves that there is always existence of a solution.

The problem is clearly linear.

Because the problem is linear we just need to establish the boundedness

$$\|u\|_U = \max_{x \in D}(L^2|\partial_{xx} u| + L|\partial_x u| + |u(x)|) = \max_{x \in D} L|\tfrac{\beta - \alpha}{L}| + |\beta \tfrac{x}{L} + \alpha \tfrac{L - x}{L}|$$

The function $\beta \frac{x}{L} + \alpha \frac{L - x}{L}$ is linear. Its maximum is either at $x = 0$ or at $x = L$. Hence

$$\|u\|_U \leq L|\tfrac{\beta - \alpha}{L}| + \max(|\beta|, |\alpha|) \leq 2 \max(|\beta|, |\alpha|) + \max(|\beta|, |\alpha|) \leq 3 \max(|\beta|, |\alpha|).$$

Hence, we have

$$\|u\|_U \leq 3 \|(\alpha, \beta)\|_G.$$

Note that uniqueness follows from the boundedness.

In conclusion, the problem is well-posed in Hadamard's sense: there is existence, uniqueness and stability (i.e., boundedness implies stability because the problem is linear).

**Question 6:** Assume that the following equation has a smooth solution: $-\partial_x\big((3+\cos(x))\partial_x T(x)\big) + 2\partial_x T(x) + T(x) = \sin(x)$, $T(a) = \sqrt{5}$, $T(b) = \pi$, $x \in (a,b)$, $t > 0$, where $b > a$. Prove that this solution is unique by using the energy method. (Hint: Do not try to simplify the expression $-\partial_x\big((3+\cos(x))\partial_x T\big)$.

Assume that there are two solutions $T_1$ and $T_2$. Let $\phi = T_2 - T_1$. Then

$$-\partial_x\big((3+\cos(x))\partial_x\phi(x)\big) + 2\partial_x\phi(x) + \phi(x) = 0, \quad \phi(a) = 0, \quad \phi(b) = 0$$

Multiply the PDE by $\phi$, integrate over $(a,b)$, and integrate by parts (i.e. apply the fundamental theorem of calculus):

$$\begin{aligned} 0 &= \int_a^b \big( -\partial_x\big((3+\cos(x))\partial_x\phi(x)\big)\phi(x) + 2(\partial_x\phi(x))\phi(x) + (\phi(x))^2\big)\mathrm{d}x \\ &= \int_a^b \big((3+\cos(x))(\partial_x\phi(x))^2 + \partial_x(\phi(x)^2) + (\phi(x))^2\big)\mathrm{d}x \\ &= \int_a^b \big((3+\cos(x))(\partial_x\phi(x))^2 + (\phi(x))^2\big)\mathrm{d}x \geq \int_a^b \big((\partial_x\phi(x))^2 + (\phi(x))^2\big)\mathrm{d}x \end{aligned}$$

because $3+\cos(x) \geq 2 > 0$ for all $x \in (a,b)$. This implies $\int_a^b (\phi(x))^2\mathrm{d}x = 0$, i.e. $\phi = 0$, meaning that $T_2 = T_1$.

**Question 7:** Assume that the following equation has a smooth solution: $-\partial_x((1+\sin(x)^2)\partial_x T(x)) - \pi\partial_x T(x) + (1+b-x)T(x) = \cos(x)$, $x \in (a,b)$, $t > 0$, with $T(a) = 1$, $T(b) = \pi$, $a < b$. Prove that this solution is unique by using the energy method. (Hint: Do not try to simplify $-\partial_x((1+\sin(x)^2)\partial_x T)$.

Assume that there are two solutions $T_1$ and $T_2$. Let $\phi = T_2 - T_1$. Then

$$-\partial_x((1+\sin(x)^2)\partial_x\phi(x)) - \pi\partial_x\phi(x) + (1+b-x)\phi(x) = 0, \quad \phi(a) = 0, \quad \phi(b) = 0$$

Multiply the PDE by $\phi$, integrate over $(a,b)$, and integrate by parts (i.e. apply the fundamental theorem of calculus):

$$\begin{aligned} 0 &= \int_a^b \big( -\partial_x\big((1+\sin(x)^2)\partial_x\phi(x)\big)\phi(x) - \pi(\partial_x\phi(x))\phi(x) + (1+b-x)(\phi(x))^2\big)\mathrm{d}x \\ &= -[\phi(x)(1+\sin(x)^2)\partial_x\phi(x)]_a^b + \int_a^b \big((1+\sin(x)^2)(\partial_x\phi(x))^2 - \pi\partial_x\Big(\frac{1}{2}\phi(x)^2\Big) + (1+b-x)(\phi(x))^2\big)\mathrm{d}x. \end{aligned}$$

But $\phi(a) = T_2(a) - T_1(a) = 0$ and $\phi(b) = T_2(b) - T_1(b) = 0$. Hence,

$$0 = \int_a^b \big((1+\sin(x)^2)(\partial_x\phi(x))^2 + (1+b-x)(\phi(x))^2\big)\mathrm{d}x \geq \int_a^b \phi^2(x)\mathrm{d}x,$$

since $1+b-x \geq 1$ for all $x \in [a,b]$. This implies $\int_a^b(\phi(x))^2\mathrm{d}x = 0$, i.e., $\phi = 0$, meaning that $T_2 = T_1$.