HW 7

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1: State the shift lemma. (Do not prove it.)

The following holds true for all integrable functions over $\mathbb{R}$ (i.e., $f \in L^1(\mathbb{R})$), and all real numbers $\beta \in \mathbb{R}$:

$$\mathcal{F}(f(x-\beta))(\omega) = \mathcal{F}(f)(\omega)e^{i\omega\beta}, \qquad \forall \omega \in \mathbb{R}.$$

Question 2: Use the Fourier transform technique to solve the following PDE:

$$\partial_t u(x,t) + \cos(t)\partial_x u(x,t) + \sin(t)u(x,t) = 0 \quad \forall x \in \mathbb{R}, t > 0, \text{ with } u(x,0) = u_0(x) \quad \forall x \in \mathbb{R}.$$

Applying the Fourier transform to the equation gives

$$\partial_t \mathcal{F}(u)(\omega,t) + \cos(t)(-i\omega)\mathcal{F}(u)(\omega,t) + \sin(t)\mathcal{F}(u)(\omega,t) = 0$$

This can also be re-written as follows:

$$\frac{\partial_t \mathcal{F}(u)(\omega,t)}{\mathcal{F}(u)(\omega,t)} = i\omega\cos(t) - \sin(t).$$

Then applying the fundamental theorem of calculus over the interval $(0,t)$, we obtain

$$\log(\mathcal{F}(u)(\omega,t)) - \log(\mathcal{F}(u)(\omega,0)) = i\omega\sin(t) + \cos(t) - 1.$$

This implies

$$\mathcal{F}(u)(\omega,t) = \mathcal{F}(u_0)(\omega)e^{i\omega\sin(t)}e^{\cos(t)-1}.$$

Then the shift lemma gives

$$\mathcal{F}(u)(\omega,t) = \mathcal{F}(u_0(x-\sin(t)))(\omega)e^{\cos(t)-1}.$$

We finally obtain

$$u(x,t) = u_0(x-\sin(t))e^{\cos(t)-1}.$$

Question 3: Use the Fourier transform technique to solve the following PDE:

$$\partial_t u(x,t) - \partial_{xx}u(x,t) + \cos(t)\partial_x u(x,t) + (1+2t)u(x,t) = 0 \quad x \in \mathbb{R}, t > 0, \qquad \text{with } u(x,0) = u_0(x).$$

(Hint: use the definition $\mathcal{F}(f)(\omega) := \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)\mathrm{e}^{i\omega x}\mathrm{d}x$; the result $\mathcal{F}(\mathrm{e}^{-\alpha x^2})(\omega) = \frac{1}{\sqrt{4\pi\alpha}}\mathrm{e}^{-\frac{\omega^2}{4\alpha}}$; the convolution theorem and the shift lemma. Go slowly and give all the details.)

Applying the Fourier transform to the equation gives

$$\partial_t \mathcal{F}(u)(\omega,t) + \omega^2 \mathcal{F}(u)(\omega,t) + \cos(t)(-i\omega)\mathcal{F}(u)(\omega,t) + (1+2t)\mathcal{F}(u)(\omega,t) = 0$$

This can also be re-written as follows:

$$\frac{\partial_t \mathcal{F}(u)(\omega,t)}{\mathcal{F}(u)(\omega,t)} = -\omega^2 + i\omega\cos(t) - (1+2t).$$

Then applying the fundamental theorem of calculus over the interval $(0,t)$, we obtain

$$\log(\mathcal{F}(u)(\omega,t)) - \log(\mathcal{F}(u)(\omega,0)) = -\omega^2 t + i\omega\sin(t) - (t+t^2).$$

This implies

$$\mathcal{F}(u)(\omega,t) = \mathcal{F}(u_0)(\omega)\mathrm{e}^{-\omega^2 t}\mathrm{e}^{i\omega\sin(t)}\mathrm{e}^{-(t+t^2)}.$$

Using the result $\mathcal{F}(\mathrm{e}^{-\alpha x^2})(\omega) = \frac{1}{\sqrt{4\pi\alpha}}\mathrm{e}^{-\frac{\omega^2}{4\alpha}}$ where $\alpha = \frac{1}{4t}$, this implies that

$$\begin{aligned}\mathcal{F}(u)(\omega,t) &= \sqrt{\frac{\pi}{t}}\mathcal{F}(u_0)(\omega)\mathcal{F}(\mathrm{e}^{-\frac{x^2}{4t}})(\omega)\mathrm{e}^{i\omega\sin(t)}\mathrm{e}^{-(t+t^2)} \\ &= \sqrt{\frac{\pi}{t}}\mathcal{F}(u_0 * \mathrm{e}^{-\frac{x^2}{4t}})(\omega)\mathrm{e}^{i\omega\sin(t)}\mathrm{e}^{-(t+t^2)}.\end{aligned}$$

Then setting $g = u_0 * \mathrm{e}^{-\frac{x^2}{4t}}$ the convolution theorem followed by the shift lemma gives

$$\mathcal{F}(u)(\omega,t) = \frac{1}{2\pi}\sqrt{\frac{\pi}{t}}\mathcal{F}(g(x-\sin(t)))(\omega)\mathrm{e}^{-(t+t^2)}.$$

This finally gives

$$u(x,t) = \sqrt{\frac{1}{4\pi t}}g(x-\sin(t))\mathrm{e}^{-(t+t^2)} = e^{-(t+t^2)}\sqrt{\frac{1}{4\pi t}}\int_{-\infty}^{+\infty} u_0(y)\mathrm{e}^{-\frac{(x-\sin(t)-y)^2}{4t}}\mathrm{d}y.$$

Question 4: Solve the following integral equation. (Hint: $(a+b)^2 = a^2 + 2ab + b^2$):

$$\int_{-\infty}^{+\infty} f(y)f(x-y)\mathrm{d}y - 2\int_{-\infty}^{+\infty} \frac{2}{y^2+1} f(x-y)\mathrm{d}y + 2\pi\frac{4}{x^2+4} = 0 \qquad \forall x \in \mathbb{R}.$$

This equation can be re-written using the convolution operator:

$$f * f - 2(\frac{2}{x^2+1}) * f + 2\pi\frac{4}{x^2+4} = 0.$$

We take the Fourier transform and use the convolution theorem to obtain

$$\begin{aligned} 2\pi\mathcal{F}(f)^2 - 4\pi\mathcal{F}(f)e^{-|\omega|} + 2\pi e^{-2|\omega|} &= 0 \\ \mathcal{F}(f)^2 - 2\mathcal{F}(f)e^{-|\omega|} + e^{-2|\omega|} &= 0 \\ (\mathcal{F}(f) - e^{-|\omega|})^2 &= 0 \end{aligned}$$

This implies

$$\mathcal{F}(f) = e^{-|\omega|}.$$

Taking the inverse Fourier transform, we obtain

$$f(x) = \frac{2}{x^2+1}.$$

Question 5: Solve the following integral equation:

$$\int_{-\infty}^{+\infty} f(y)f(x-y)\mathrm{d}y - 2\sqrt{2}\int_{-\infty}^{+\infty} e^{-\frac{y^2}{2\pi}} f(x-y)\mathrm{d}y = -2\pi e^{-\frac{x^2}{4\pi}} \qquad \forall x \in \mathbb{R}.$$

This equation can be re-written using the convolution operator:

$$f * f - 2\sqrt{2}e^{-\frac{x^2}{2\pi}} * f = -2\pi e^{-\frac{x^2}{4\pi}}.$$

We take the Fourier transform and use the convolution theorem and obtain

$$\begin{aligned} 2\pi\mathcal{F}(f)^2 - 2\pi 2\sqrt{2}\mathcal{F}(f)\frac{1}{\sqrt{4\pi\frac{1}{2\pi}}}e^{-\omega^2\frac{1}{4\frac{1}{2\pi}}} &= -2\pi\frac{1}{\sqrt{4\pi\frac{1}{4\pi}}}e^{-\omega^2\frac{1}{4\frac{1}{4\pi}}} \\ \mathcal{F}(f)^2 - 2\mathcal{F}(f)e^{-\omega^2\frac{\pi}{2}} + e^{-\omega^2\pi} &= 0 \\ (\mathcal{F}(f) - e^{-\omega^2\frac{\pi}{2}})^2 &= 0. \end{aligned}$$

This implies

$$\mathcal{F}(f) = e^{-\omega^2\frac{\pi}{2}}.$$

Taking the inverse Fourier transform, we obtain

$$f(x) = \sqrt{2}e^{-\frac{x^2}{2\pi}}.$$

Question 6: Let $\alpha \geq 0$ and $c > 0$. Let $g \in L^1(\mathbb{R})$. Consider the telegraph equation

$$\partial_{tt} u + 2\alpha \partial_t u + \alpha^2 u - c^2 \partial_{xx} u = 0, \quad u(x,0) = 0, \quad \partial_t u(x,0) = g(x) \quad \forall x \in \mathbb{R}, t > 0,$$

and boundary condition at infinity $u(\pm\infty, t) = 0$. Solve the equation by the Fourier transform technique. Hint: the solution to the ODE $\phi''(t) + 2\alpha\phi'(t) + (\alpha^2 + \lambda^2)\phi(t) = 0$ is $\phi(t) = \mathrm{e}^{-\alpha t}(a\cos(\lambda t) + b\sin(\lambda t))$ and the Fourier transform of

$$S_\lambda(x) = \begin{cases} 1 & \text{if } |x| \leq \lambda \\ 0 & \text{otherwise} \end{cases} \qquad \text{is} \qquad \mathcal{F}(S_\lambda)(\omega) = \frac{1}{\pi}\frac{\sin(\lambda\omega)}{\omega} \quad \forall \omega \in \mathbb{R}, \omega \neq 0. \tag{1}$$

Applying the Fourier transform with respect to x to the equation, we infer that

$$\begin{aligned} 0 &= \partial_{tt}\mathcal{F}(u)(\omega,t) + 2\alpha\partial_t\mathcal{F}(u)(\omega,t) + \alpha^2\mathcal{F}(u)(\omega,t) - c^2(-i\omega)^2\mathcal{F}(u)(\omega,t) \\ &= \partial_{tt}\mathcal{F}(u)(\omega,t) + 2\alpha\partial_t\mathcal{F}(u)(\omega,t) + (\alpha^2 + c^2\omega^2)\mathcal{F}(u)(\omega,t) \end{aligned}$$

Using the hint, we deduce that

$$\mathcal{F}(u)(\omega,t) = \mathsf{e}^{-\alpha t}(a(\omega)\cos(\omega ct) + b(\omega)\sin(\omega ct)).$$

The initial condition implies that $a(\omega) = 0$ and $\mathcal{F}(g)(\omega) = \omega c b(\omega)$; as a result, $b(\omega) = \mathcal{F}(g)(\omega)/(\omega c)$ and

$$\mathcal{F}(u)(\omega,t) = \mathsf{e}^{-\alpha t}\mathcal{F}(g)\frac{\sin(\omega ct)}{\omega c}.$$

Then using (1), we have

$$\mathcal{F}(u)(\omega,t) = \frac{\pi}{c}\mathsf{e}^{-\alpha t}\mathcal{F}(g)\mathcal{F}(S_{ct}).$$

The convolution theorem implies that

$$u(x,t) = \mathsf{e}^{-\alpha t}\frac{1}{2c}g * S_{ct} = \mathsf{e}^{-\alpha t}\frac{1}{2c}\int_{-\infty}^{\infty} g(y)S_{ct}(x-y)\mathsf{d}y.$$

Finally the definition of S_{ct} implies that $S_{ct}(x-y)$ is equal to 1 if $-ct < x - y < ct$ and is equal zero otherwise, which finally means that

$$u(x,t) = \mathsf{e}^{-\alpha t}\frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\mathsf{d}y.$$
